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Random Processes With Specified Spectral Density and First-Order Probability Density

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The procedure for generating a Gaussian process with a specified spectral density is well known. It is harder to generate a process with a specified spectral density and a specified first-order probability distribution. In this paper we explore, by simulation, the possibility of generating a process with such a dual specification by passing a Gaussian process with an appropriately chosen spectral density through an appropriately chosen zero-memory nonlinearity. Several applications are cited where such a dual specification is desirable.

I. INTRODUCTION

The procedure for generating a Gaussian random process whose power spectral density (psd) is a specified function of frequency, $S(\omega)$, is well known. As we see in Fig. 1, let $H(j\omega)$ be the transfer function of a linear time-invariant filter, and let the input to the filter be a Gaussian random process $\{x(t)\}$ with psd $\Phi_x(\omega)$. Then the psd of the output process $\{y(t)\}$ is given by

$$\Phi_y(\omega) = H(j\omega)H^*(j\omega)\Phi_x(\omega) \tag{1}$$

where * denotes complex conjugation. Since the filter H is linear, $\{y\}$ is also a Gaussian random process, and if $\{x\}$ is a white noise[†] with unit psd, then $\{y\}$ has the desired psd provided

$$H(j\omega)H^*(j\omega) = S(\omega). \tag{2}$$

H is then called a shaping filter (see Ref. 1) for the psd $S(\omega)$. The spectral factorization for eq. (2) can be accomplished analytically when

[†] A high degree of mathematical rigor is not intended here. For our purpose we define white noise as a noise whose psd is constant over a wide bandwidth $(-W, W)$ and zero outside. The bandwidth is assumed wide enough so that any desired psd is negligible outside it.

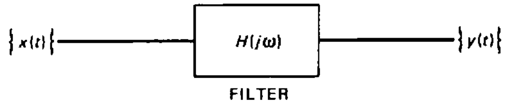

Fig. 1—Generating a Gaussian random process with specified-power spectral density.

$S(\omega)$ is a rational function of ω^2.[1] In such cases the resulting shaping filter too can be synthesized by a standard procedure as a lumped parameter filter (see Ref. 2). The appropriate shaping filter can of course be derived for more general spectral densities as well as for various nonstationary processes. Numerical approximations to the shaping filter can be obtained for any reasonably well-behaved spectral density, as we shall see in Section III.

This method of generating the random process, however, leaves no choice as to the first-order probability density function (pdf) of the process, i.e., the pdf of the random variable $Y = y(t)$. Since $\{y(t)\}$ is a Gaussian process, all joint probabilities of the random variables $Y_i = y(t_i)$, $i = 1, 2, \cdots$ are Gaussian. Hence, the pdf of Y too is Gaussian. (Of course, if *only* the pdf is specified, then one does not have to follow the above procedure. It is always possible to generate a white noise with any specified pdf by passing, for instance, a uniform white noise or a Gaussian white noise through a zero-memory nonlinearity. The procedure is quite analogous to the one discussed in Section 2.1.)

Frequently it is desirable to specify both the spectral density and the first-order pdf of the process. One situation where such a specification would be useful is in the simulation of speech-waveform coders. The performance of such coders can depend significantly upon both the spectral density and the pdf of the signal to be coded. Measurements have shown[3] that the pdf for speech signals is markedly different from Gaussian, and is in fact much better represented by a "gamma" distribution[4] (see Section II). At present, simulations with such coders are carried out on Gaussian processes with appropriately shaped spectra, or on sequences of uncorrelated samples with a gamma pdf. The behavior of the coders on speech signals is not well predicted by either of these; hence, tests are also performed on a variety of speech sentences. Perhaps these tests could be standardized and their predictive value improved by the use of random processes with a gamma pdf and a selection of typical spectral shapes.

Another area where this dual specification can be important is in perceptual studies. One such application is to Julesz's experiments on texture perception.[5] The independent control of spectral density and pdf of random dot patterns would enable one to decide between competing theories of texture discrimination.

Finally, such control of pdf and spectral density would be useful in studying input-output properties of nonlinear systems, which can be represented by a zero-memory nonlinearity followed by a linear filter.

The problem of synthesizing a random process to approximate the pdf and power spectral density of a given process has been addressed in the literature.[6] However, to the best of our knowledge no exact procedure is known at present for generating a random process with specified pdf and spectral density. In the next two sections we explore, by simulation, the capabilities and limitations of one simple attack on the problem.

II. GENERATION OF THE RANDOM PROCESS

The method of generation that we wish to study is shown schematically in Fig. 2. For simplicity we will assume throughout that the desired random process has zero mean and that the desired pdf has even symmetry. These restrictions are by no means essential for our analysis, but only simplify our presentation.

The basic idea of the proposal is as follows: We start with a "white" Gaussian noise [see the footnote in Section I] and pass it through a filter $H(j\omega)$ and a scale factor K such that the random process $\{x(t)\}$ is a zero-mean-Gaussian process with unit variance. Let $q(.)$ be the desired pdf of the random variable $Y = y(t)$. Then it is straightforward to find the zero-memory nonlinearity $F(.)$ such that Y has the desired pdf. The problem then is to find H such that *after the nonlinear distortion by* $F(.)$ the spectral density at the output is the desired function $S(\omega)$. It is easy to come up with examples for which this problem has no solution. For instance it can be shown that $\{v(t)\}$ cannot be a strictly band-limited process for *any* choice of limiting nonlinearity $F(.)$.[7] Nevertheless, as we will show, in a variety of cases of interest the problem has a solution, or an approximate solution.

Before proceeding to a detailed description of the method, we may emphasize the reason for the choice of a Gaussian process for the input. This is the property of the Gaussian process that it stays Gaussian after a linear transformation. The Gaussian process is the only well-behaved process that has this property. The same reason also dictates the order of operations in Fig. 2. Thus, interchanging the filter and zero-memory nonlinearity would be equivalent to generating the output process by a linear transformation of a non-Gaussian white

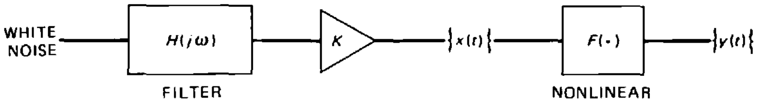

Fig. 2—Method for generating a random process with specified-power spectral density *and* specified first-order probability density.

noise. While it is possible to generate processes with non-Gaussian pdf's this way, it is not easy to compute (or control) the pdf of the output process.

2.1 Derivation of the function F(.)

Let us begin by deriving the required function $F(.)$. It is convenient to think of $F(.)$ as a composition of two functions, $P(.)$ and $f(.)$, as shown in Fig. 3. In view of the assumptions about the process $\{x(t)\}$, the random variable $X = x(t)$ has the pdf

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}. \tag{3}$$

If $\hat{X} = P(X)$ where $P(x)$ is the cumulative distribution

$$P(x) = \int_{-\infty}^{x} p(\lambda) d\lambda, \tag{4}$$

then the pdf of the random variable $\hat{X}$ is uniform on the interval (0, 1). Similarly, if $Y = f(\hat{X})$ and Y has the desired pdf $q(y)$, then the cumulative distribution $Q(y)$ is related to the function f^{-1} by the equation

$$Q(y) = \int_{-\infty}^{y} q(\lambda) d\lambda = f^{-1}(y). \tag{5}$$

Thus, in order for Y to have the desired pdf, $f(.)$ must be chosen to

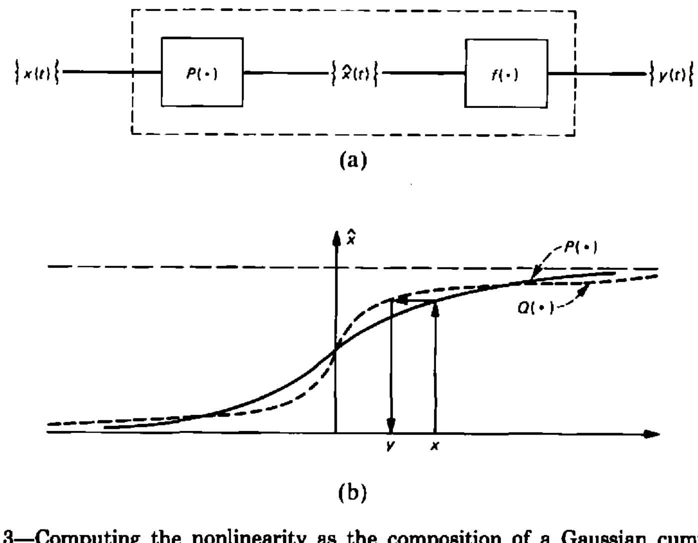

Fig. 3—Computing the nonlinearity as the composition of a Gaussian cumulative probability and the inverse of the desired cumulative distribution.

satisfy eq. (5). As long as $q(y)$ is not identically zero over an interval, Q is an invertible function and eq. (5) defines f. The technical difficulty arising when this condition is not met is resolved in an obvious manner. Thus, if $q(y)$ is nonzero everywhere except on the interval $a \le y \le b$, then $Q(y)$ is constant from a to b. Hence, $\hat{X}$ as a function of Y has a jump at $Y = Q(a)$ of magnitude $b - a$. The limit of the function f can be defined from above and below this value. At $Q(a)$ the function may be specified arbitrarily.

Finally, the function $F(.)$ is given by

$$F(x) = f[P(x)]. \tag{6}$$

From the assumed symmetry of $q(y)$ it is evident that $F(.)$ must turn out to be an odd function of its argument.

By way of illustration in this paper we will consider three output pdf's:
the uniform pdf:

$$q(y) = \frac{1}{2\sqrt{3}}, \qquad |y| \le \sqrt{3}; \tag{7a}$$

the gamma* pdf

$$q(y) = \frac{3^{1/4}}{\sqrt{8\pi}} \frac{1}{\sqrt{|x|}} e^{-|x|\sqrt{3}/2}; \tag{7b}$$

and the binary pdf

$$q(y) = ½[\delta(y - 1) + \delta(y + 1)]. \tag{7c}$$

In each case the constants have been chosen to normalize the variance of the pdf to be 1. Our choice of examples is of course arbitrary. However, the uniform and the binary are obvious simple examples one expects to be useful, and the gamma has the relevance to speech signals mentioned above. The function $F(.)$ turns out to be quite simple for each of these cases:
for the uniform pdf

$$F(x) = 2\sqrt{3}[P(x) - ½], \tag{8a}$$

for the gamma pdf

$$F(x) = \frac{1}{\sqrt{3}} x^2 \text{sign}(x), \tag{8b}$$

and for the binary pdf

$$F(x) = \text{sign}(x). \tag{8c}$$

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* Strictly speaking, a gamma 1/2 distribution.

2.2 The covariance function of $\{y\}$

Having derived the nonlinearity F, we must now find the relationship between the spectral properties of $\{x\}$ and $\{y\}$. Because of the nonlinear transformation, the relationship is best derived in the time domain, i.e., between the autocovariance functions of the two processes. [Recall that the autocovariance function (acf) is just the Fourier inverse transform of the spectral density.] Let $\rho(\tau)$ be the acf of the process $\{x\}$, i.e., $\rho(\tau) = E[x(t)x(t+\tau)]$. Let $g(u, v, \rho)$ be the unit variance, zero-mean, two-dimensional Gaussian pdf given by

$$g(u, v, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2}(u^2+v^2-2\rho uv)}. \tag{9}$$

Then the acf of the $\{y\}$ process is given by

$$R = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u)F(v)g(u, v, \rho)dudv, \tag{10}$$

where, of course, ρ (and hence R) is a function of the lag, τ. A general method of evaluating the integrals in eq. (10) is to use Mehler's expansion

$$g(u, v, \rho) = \frac{1}{2\pi} e^{-\frac{1}{2}(u^2+v^2)} \sum_0^\infty \frac{\rho^n}{n!} \mathrm{He}_n(u)\mathrm{He}_n(v) \tag{11}$$

of the two-dimensional Gaussian pdf in terms of Hermite polynomials.[8] Then if $F(.)$ has an expansion in terms of the same polynomials

$$F(x) = \sum_0^\infty f_n \mathrm{He}_n(x)/\sqrt{n!}, \tag{12}$$

it can be shown[7] that

$$R = \sum_1^\infty f_n^2 \rho^n. \tag{13}$$

When F is an odd function, as is the case here, the even coefficients in eqs. (12) and (13) vanish, and it is seen that R is an odd function of ρ. [This can of course be seen directly from eq. (10) by noting that $g(u, v, -\rho) = g(u, -v, \rho)$, and changing the sign of one of the variables of integration.]

For the examples that we have chosen, there is no need to use this general procedure. In these cases the integration is easily carried out in closed form. As shown in the appendix, the relationship between R and ρ is as follows:

$$\text{Uniform: } R = \frac{6}{\pi}\sin^{-1}\frac{\rho}{2}. \tag{14a}$$

$$\text{Gamma: } R = \frac{2}{3\pi}\left[(1 + 2\rho^2)\sin^{-1}\rho + 3\rho\sqrt{1-\rho^2}\right] \tag{14b}$$

$$\text{Binary: } R = \frac{2}{\pi}\sin^{-1}\rho. \tag{14c}$$

The three functions are shown in Fig. 4 for $0 \le \rho \le 1$; the odd symmetry gives the function for negative ρ as well.

2.3 Derivation of the Filter H

The functions in Fig. 4 are monotonic increasing functions over the whole range $-1 \le \rho \le 1$. [From eqs. (12) and (13) this property is seen to be true whenever F has odd symmetry.] Therefore, the relationship between ρ and R is invertible.

Suppose now that $R(\tau)$ is the Fourier transform of the desired spectral density $S(\omega)$. Then the function

$$\phi(\tau) = \rho[R(\tau)] \tag{15}$$

can be computed, where $\rho(R)$ is the function corresponding to the desired pdf. For the examples chosen we get

$$\text{Uniform: } \phi_u(\tau) = 2\sin\left[\frac{\pi}{6}R(\tau)\right] \tag{16a}$$

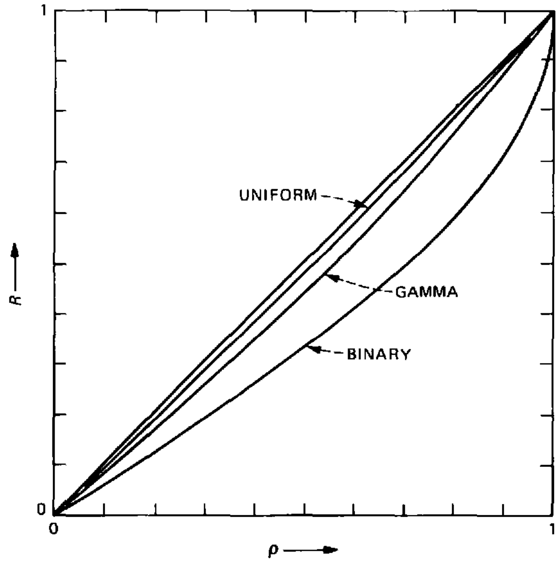

Fig. 4—The relationship between the autocovariance (R) at the output and (ρ) at the input of the nonlinearity appropriate to the generation of the three desired probability distributions. The unmarked line is $R = \rho$. Since the functions have odd symmetry, only the range of positive covariance values is shown.

$$\text{Gamma: } \phi_g(\tau) = \rho_g[R(\tau)] \tag{16b}$$

$$\text{Binary: } \phi_b(\tau) = \sin\left[\frac{\pi}{2} R(\tau)\right], \tag{16c}$$

where ρ_g is the inverse of the relation (14b). We are unable to give ρ_g explicitly; however, its numerical computation is trivial.

The function $\phi(\tau)$ is continuous and symmetric so that its Fourier transform, $\Phi(\omega)$, is real. For many specified acf's $R(\tau)$, $\Phi(\omega) \geq 0$ for all ω. In such cases we can obviously obtain an exact solution to the problem. All we need to do is to synthesize a filter, $H(j\omega)$, such that

$$K^2 H(j\omega) H^*(j\omega) = \Phi(\omega) = \mathbf{F}[\phi(\tau)], \tag{17}$$

where $\mathbf{F}$ denotes the Fourier transform. Unfortunately, $\Phi(\omega)$ is not guaranteed to be nonnegative at all frequencies for every specified acf $R(\tau)$. This is because when $\rho(\tau)$ in eq. (10) ranges over all possible acf's, $R(\tau)$ ranges only over a subset of possible acf's.

If the desired spectral density is such that $\Phi(\omega)$ becomes negative at some frequencies, the best we can do with our method is to give an approximate solution as follows: Define the function $\Phi^+(\omega)$ such that

$$\begin{aligned} \Phi^+(\omega) &= \Phi(\omega) \quad \text{if } \Phi(\omega) \geq 0 \\ &= 0 \qquad\quad \text{otherwise.} \end{aligned} \tag{18}$$

We then synthesize $H(j\omega)$ such that

$$K^2 H(j\omega) H^*(j\omega) = \Phi^+(\omega). \tag{19}$$

In the next section we will use eq. (19) [or its special case, eq. (17)] to generate random processes with a variety of pdf's and spectral densities.

III. SIMULATIONS

In this section we will describe the numerical procedures required to generate a random process with a pdf $q(.)$ and a spectral density $S(\omega)$, based on the theoretical discussion of the previous section. We have already shown how the nonlinearity $F(.)$ is to be computed. It remains to be shown how to numerically approximate the shaping filter.

3.1 The shaping filter

We will approximate the shaping filter as a transversal filter (FIR filter). Let $R(\tau)$ be the desired acf. Unless $R(\tau)$ happens to be of finite duration, it must first be truncated. To ensure that the truncated function $\hat{R}(\tau)$ is a legitimate acf, the truncation must be done by multiplying the given acf by an acf of finite duration. (Recall that the product of two acf's is an acf). We chose a Hamming window convolved with itself as the truncating window. Thus let

$$w(\tau) = \int_{-\infty}^{\infty} h(t)h(t-\tau)dt, \tag{20}$$

where $h(t)$ is the Hamming window defined as

$$h(t) = 0.54 + 0.46 \cos\frac{\pi|t|}{T}, \ |t| \leq T \tag{21}$$

$$= 0 \qquad \text{otherwise.}$$

Then we define the truncated acf as

$$\hat{R}(\tau) = w(\tau)R(\tau). \tag{22}$$

For $R(\tau)$ decaying rapidly enough as $\tau \to \infty$, $\hat{R}(\tau)$ can be made to approximate $R(\tau)$ as closely as desired by choosing T sufficiently large. For the rest of the paper, therefore, we will regard $\hat{R}(\tau)$ as the desired acf, and $\hat{S}(\omega) = \mathbf{F}[\hat{R}(\tau)]$ as the desired spectral density.

The filter is synthesized from $\hat{R}(\tau)$ by the following sequence of steps:

(*i*) Discretize $\hat{R}(\tau)$ to give the sequence $\hat{R}_n$, $-N + 1 \leq n \leq N$.*

(*ii*) Compute the sequence $\phi_n = \phi(\hat{R}_n)$, where ϕ is the function defined by eq. (15) for the appropriate desired pdf.

(*iii*) Find the FFT of the sequence ϕ_n and set any negative values to zero. Let μ_n denote the resulting, adjusted Fourier transform.

(*iv*) The desired impulse response is then the inverse FFT of the sequence $\sqrt{\mu_n}$.

3.2 Examples

For each of the pdf's (a), (b), and (c) we have generated several examples of random processes with acf's selected from the following types:

$$R(\tau) = e^{-\alpha\tau} \tag{23a}$$

$$R(\tau) = \sum_{1}^{4} a_i e^{-\alpha_i\tau}\cos(\beta_i\tau), \tag{23b}$$

where $a_i \geq 0$ in the last equation. By proper choice of parameters in eq. (23b), the spectrum can be made to approximate that of any vowel sound up to about 4 kHz. The β_i's are the formant frequencies and the α_i's the half bandwidths.

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* The asymmetry of the range of values for n makes the number of values even. Choosing N to be a power of 2 allows the use of efficient FFT (Fast Fourier Transform) routines.

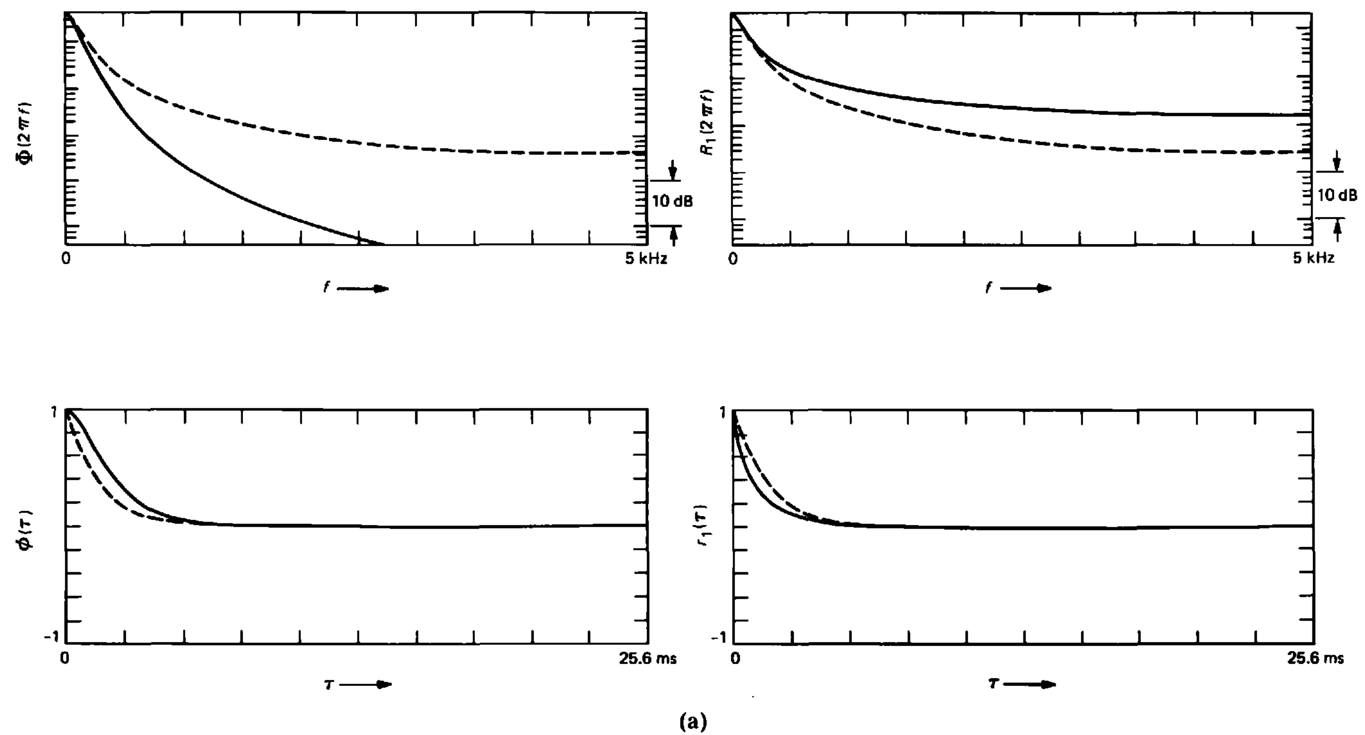
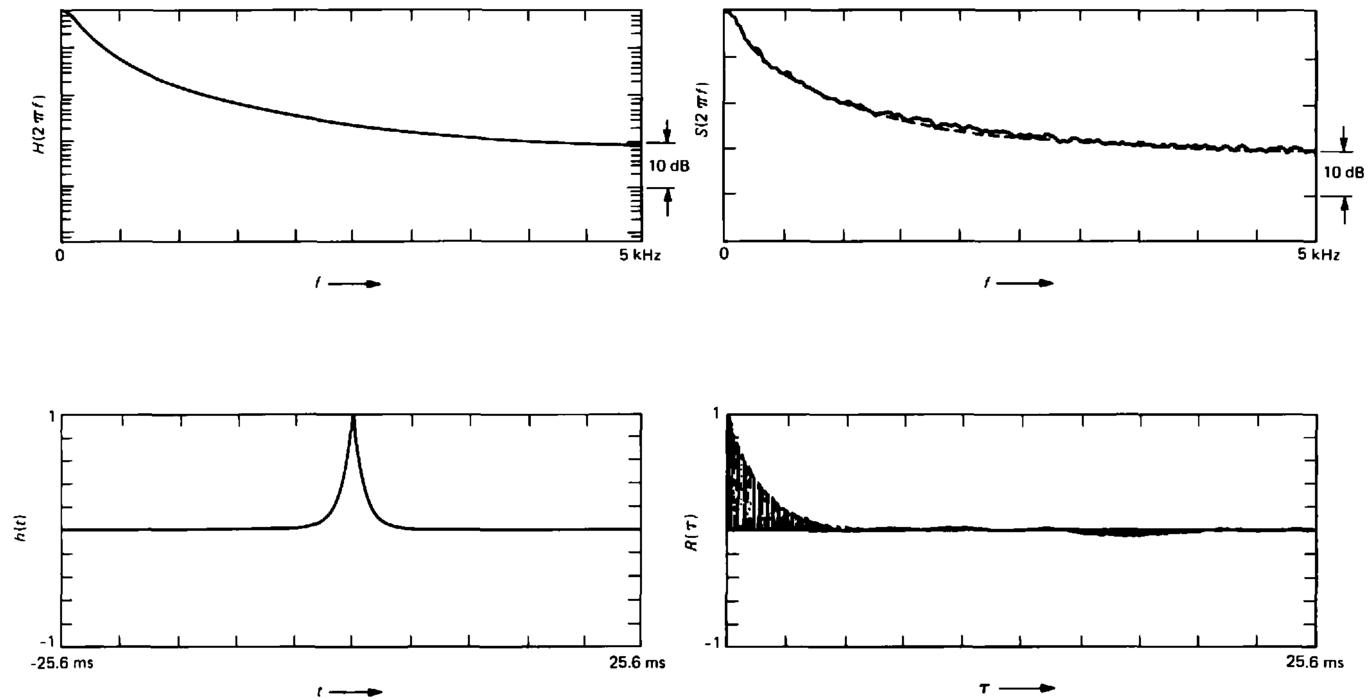


Fig. 5a—Generating a binary process with an exponential acf with 100-Hz cutoff frequency. On each graph the dotted line is the desired or prescribed function for the output process. The acf and spectral density required for the $\{x\}$ process are $\phi(\tau)$ and $\Phi(\omega)$, respectively. $r_1(\tau)$ and $R_1(\omega)$ are what the corresponding functions would be for the output process $\{y\}$ if the input $\{x\}$ had the specified acf.



(b)

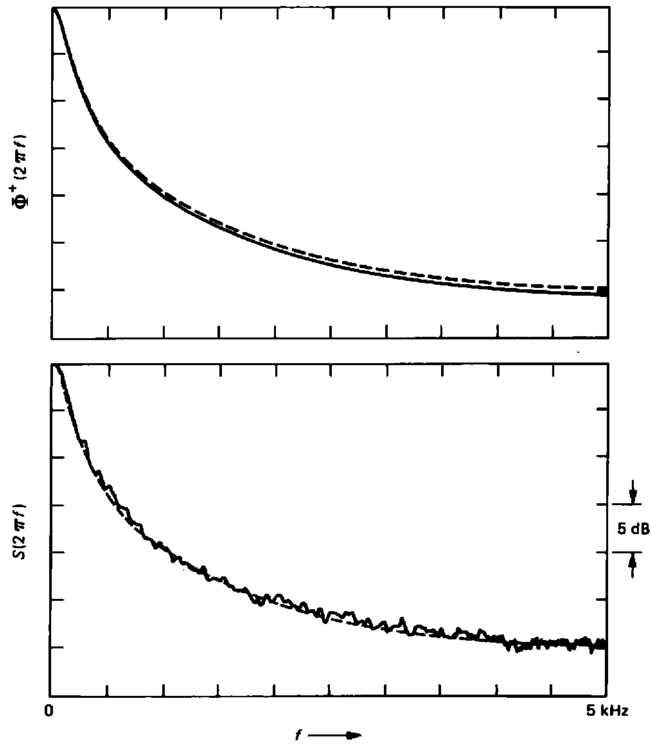
Fig. 5b—On the left are shown the time and frequency domain plots of the shaping filter required to produce the $\{x\}$ process of Fig. 5a. On the right are the acf and spectral density of an actual binary process generated by the method described in the text.

The white noise input to the shaping filter was just a sequence of independent, identically distributed Gaussian random variables. The Gaussian distribution was truncated to ±6 standard deviations.

In Figs. 5 through 9 we show several examples of acf's and spectra (both as predicted theoretically and as measured from the actual outputs) that can be generated by our method.

Figure 5 shows in detail various covariances and spectra associated with the generation of a binary process with an exponential covariance function. Whenever a dotted graph is displayed it is the desired or specified function.

In Fig. 5a the left side shows the acf and spectral density $\phi(\tau)$ and $\Phi(\omega)$ of the $\{x\}$ process, which when passed through the clipping nonlinearity gives the process $\{y\}$ with the specified properties. The right side of Fig. 5a shows the same plots for the process $\{y\}$ that would result if $\{x\}$ were a Gaussian process that had the *specified* acf.



(a)

Fig. 6a—On the top the spectral density $\Phi(\omega)$ required for the $\{x\}$ process and on the bottom the spectral density estimated from a 25,000-sample sequence. The spectral density specified is the same as in Fig. 5.

In Fig. 5b the left side shows the time and frequency domain plots of the shaping filter to produce the required $\{x\}$, and the right side shows the acf and spectral density estimated from a 25,000-sample sequence generated by the method discussed above for the binary case.

In Figs. 6 through 9 we show spectral density plots only. Further, we show only the plot for $\Phi^+(\omega)$, as defined in eq. (19), and the plot of the output spectral density estimated from 25,000-sample sequences generated by our method. (Of course, Φ^+ is often identical to Φ.) Again, in each case the dotted graphs represent the desired or specified function. The selections shown are an exponential acf for the uniform and gamma distributions in Fig. 6; spectra of vowels /a/ and /e/ with a uniform distribution in Fig. 7; the same vowels with a gamma distribution in Fig. 8; the same vowels with a binary distribution in Fig. 9.

The probability distributions of the generated processes are, of course, not approximated; they are therefore exact except for fluctuations because of finite sample size. Figure 10 shows the actual and

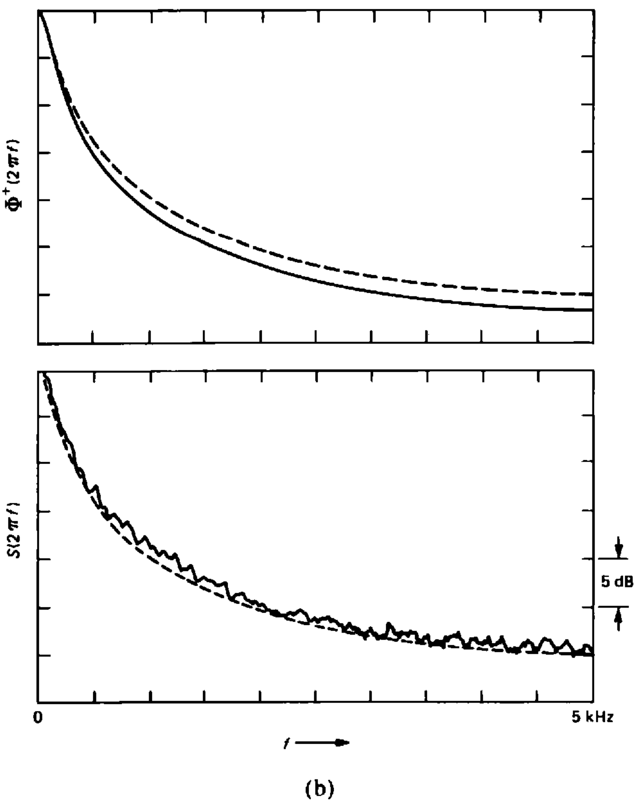
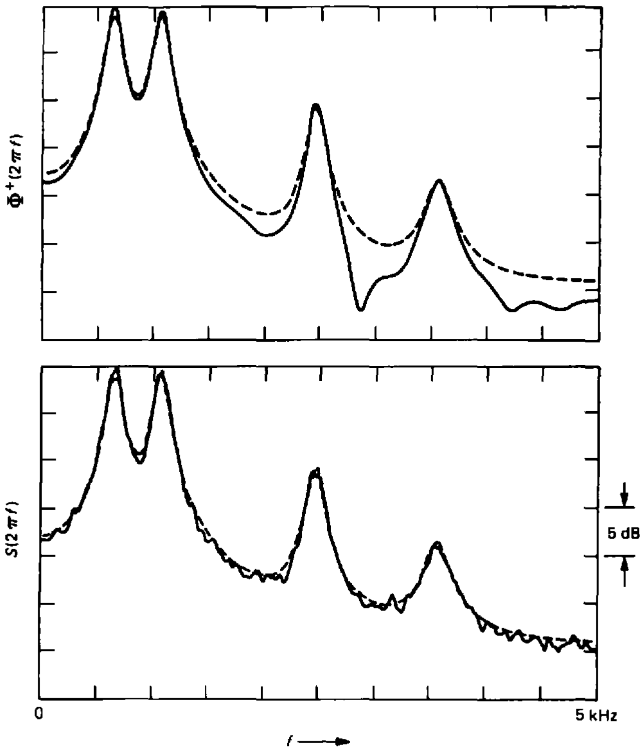


Fig. 6b—Same as Fig. 6a but for the gamma distribution.



(a)

Fig. 7a—The specified spectral density is a typical spectral shape for the vowel /a/. On the top is $\Phi(\omega)$ for the required $\{x\}$ process. On the bottom is the plot estimated from a 25,000-sample sequence of the generated process. The dotted line in each case is the specified spectral density.

expected distributions for a 25,000-sample sequence with a uniform and a gamma distribution.

Figures 5 through 10 are self-explanatory and demonstrate the capabilities and limitations of the method. We may summarize the main features as follows:

(*i*) When the specified acf is given by eq. (23a), the problem can be solved exactly for any α for each of our examples of pdf. For the uniform and the binary pdf this can be proved analytically, by expanding the sin function in eqs. (16a) and (16c) in powers of R. If we group terms of these expansions in pairs, we can show that the Fourier transform of $\phi(\tau)$ is nonnegative in each of these cases. For the gamma pdf we cannot prove this analytically; however by simulation over a wide range of α's we conclude that $\phi(\tau)$ has a nonnegative Fourier transform in this case too.

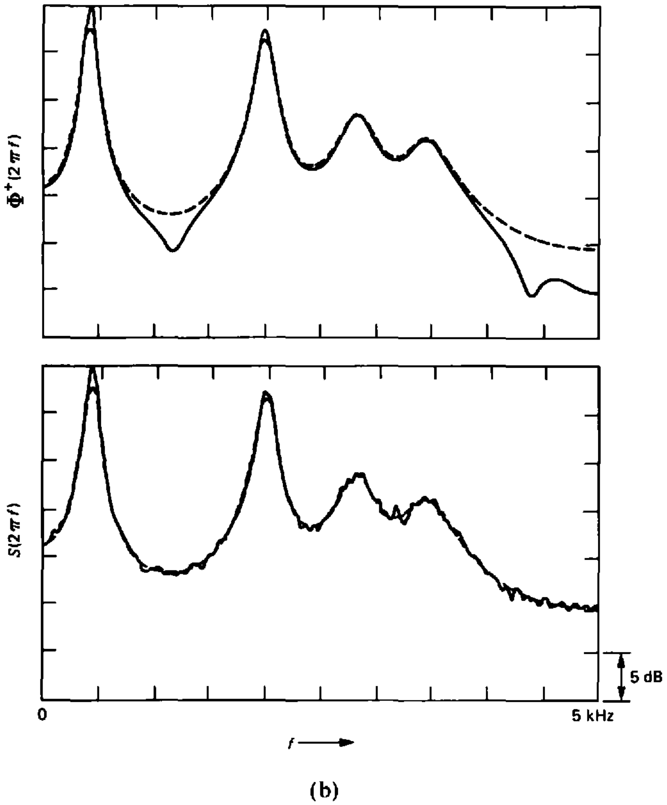


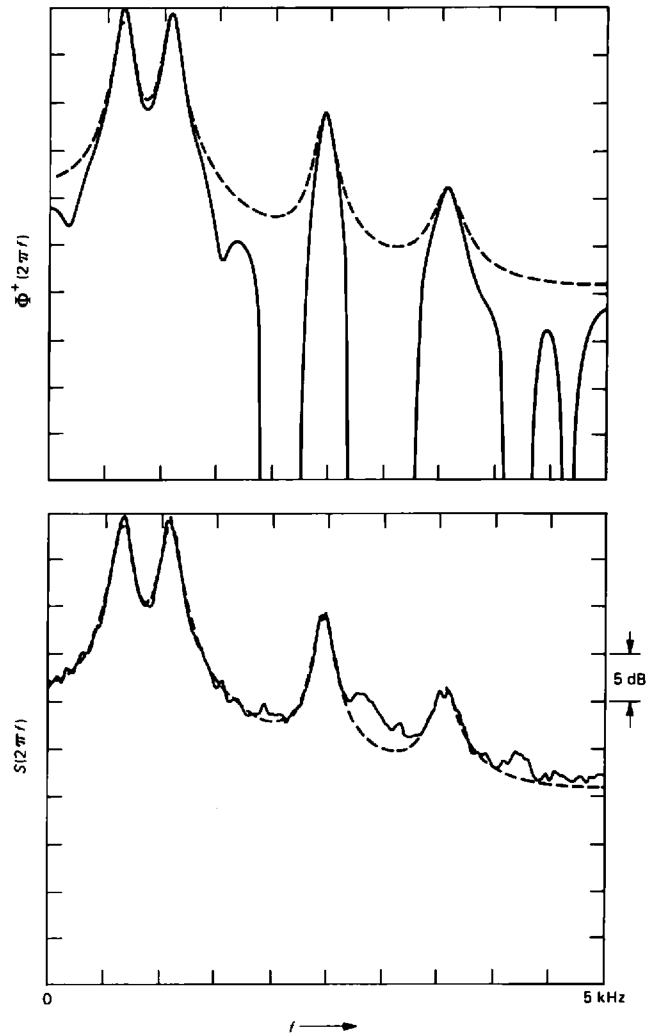
Fig. 7b—Same as Fig. 7a but for the vowel /e/. The plots are for a process with a uniform probability density.

(*ii*) Vowel spectra can be well approximated by random processes with a uniform or gamma distribution. (In the two cases shown in Figs. 7 and 8, the spectrum is realised exactly with the uniform pdf, but only approximately with the gamma pdf; however, the error in the spectrum in the latter case is not much larger than the statistical fluctuations in a 25,000-sample segment. So from a practical point of view the approximation error might not be serious.)

(*iii*) The nonlinearity required for the binary pdf is too severe to allow a well-defined formant structure, especially at high frequencies.

IV. CONCLUDING REMARKS

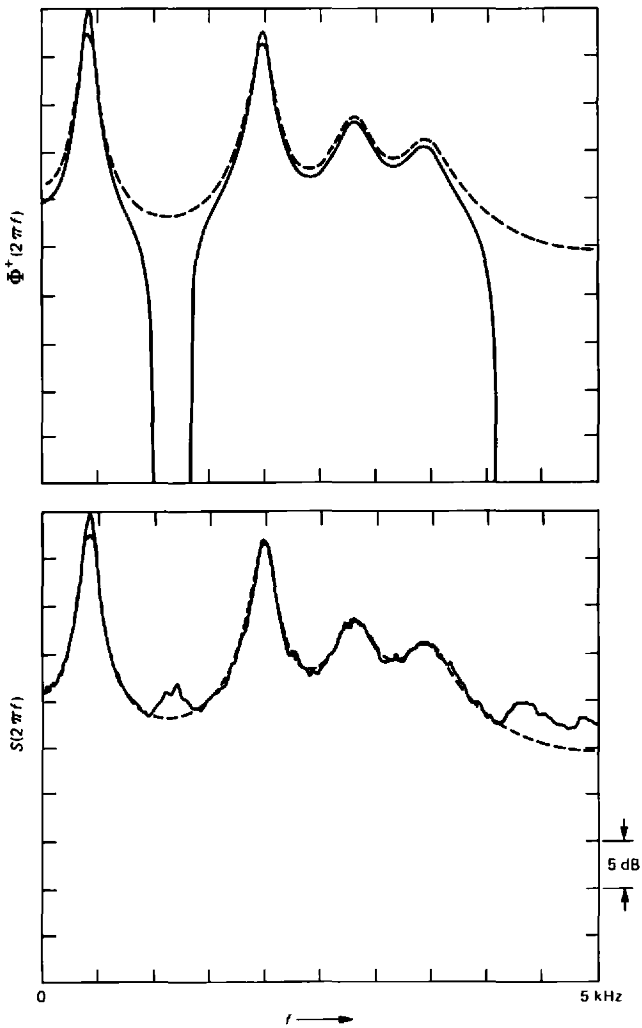
We have described a method for generating a random process with specified spectrum and first-order probability density. The method is successful for many combinations of spectrum and pdf of interest. However, with a given probability density, the method cannot achieve



(a)

Fig. 8a—The specified spectral density is a typical spectral shape for the vowel /a/ for a process with a gamma probability distribution. On the top is $\Phi(\omega)$ for the required $\{x\}$ process. On the bottom is the plot estimated from a 25,000-sample sequence of the generated process. Note that $\Phi(\omega)$ had to be corrected to $\Phi^+(\omega)$, as shown in eq. (15).

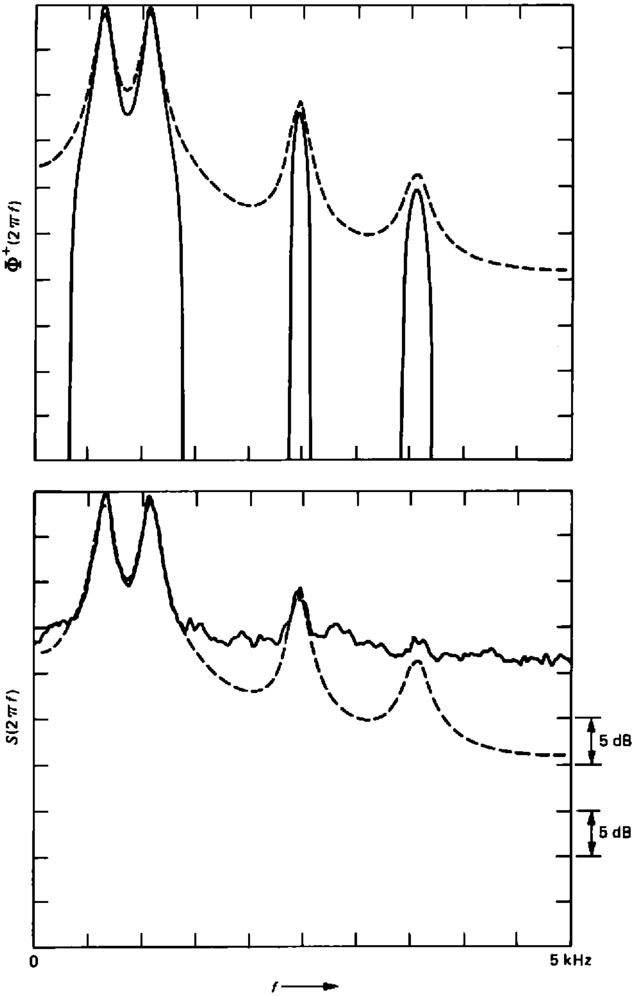
every arbitrarily specified spectral shape. One general way to characterize an achievable spectrum is to say that the corresponding covariance function must be representable in the form given in eq. (13). Another way is to say that the function $\phi(\tau)$ in eq. (15) must have a nonnegative Fourier transform.



(b)

Fig. 8b—Same as Fig. 8a but for /e/.

The class of achievable spectra may be extended by other methods of generating the random process. One interesting method has been suggested by E. N. Gilbert. Rather than nonlinearly distorting a Gaussian process, the suggestion is to *modulate* a Gaussian process by an appropriately chosen nonnegative random process. This method has its own limitations. For instance, it cannot generate a process with a uniform pdf. On the other hand it appears to be very promising for

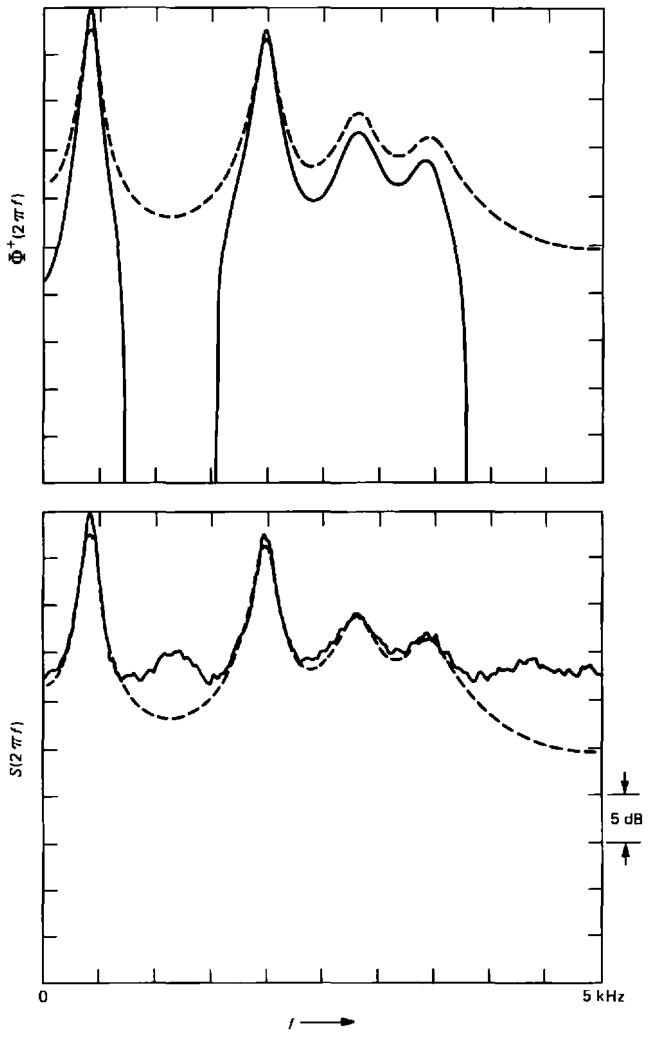


(a)

Fig. 9a—The specified spectral density is a typical spectral shape for the vowel /a/ for a process with a binary distribution. On the top is $\Phi(\omega)$ for the required $\{x\}$ process. On the bottom is the plot estimated from a 25,000-sample sequence of the generated process. Note that $\Phi(\omega)$ had to be corrected to $\Phi^{+}(\omega)$, as shown in eq. (15).

generating processes with speech-like pdf's (e.g., the gamma pdf discussed above). We are currently investigating this possibility.

To some extent the pdf and covariance function constrain each other regardless of the method used for generating the random process. For example, the covariance function of a time-continuous binary process must have a cusp at the origin. If the specified acf does not



(b)

Fig. 9b—Same as Fig. 9a but for the vowel /e/.

have this property, then the specifications are inconsistent and the problem has no solution. Unfortunately, to the best of our knowledge, no tractable procedure is known to decide whether a covariance function is consistent with a given pdf. L. A. Shepp has drawn our attention to some of his unpublished work in which he investigated the class of covariance functions of processes with a given pdf. He showed that this class is convex and that any such acf is a convex

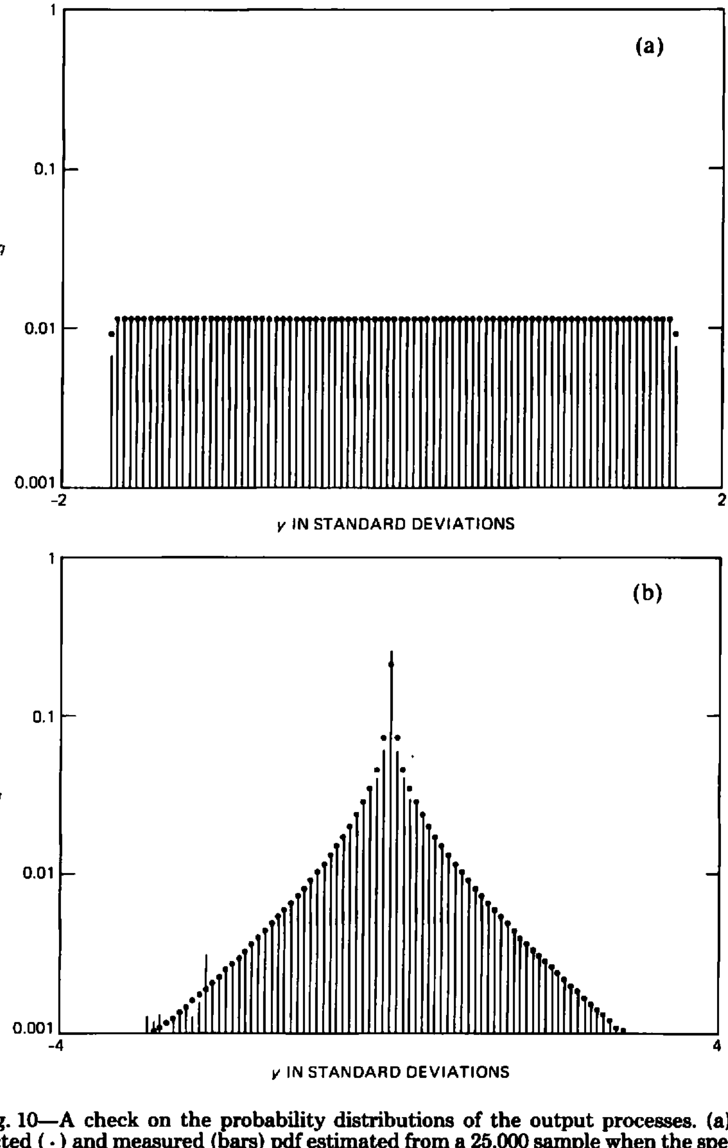


Fig. 10—A check on the probability distributions of the output processes. (a) The expected (·) and measured (bars) pdf estimated from a 25,000 sample when the specified distribution was uniform. The fit is approximately as good as this for every uniform-pdf process generated. (b) Same as for Fig. 10a but for the gamma distribution.

combination of the extremal acf's of the class. Unfortunately, no general method is known to determine the extremal covariances. Another relevant work is a paper[9] by Brockway McMillan, in which he considers covariances of binary processes, and gives a test for such a covariance in terms of the class of "corner-positive" matrices. This test too is very difficult to apply in practice.

In some situations (for example, the experiment on texture perception mentioned above) it might not be important to specify the spectrum and the pdf precisely. Instead, it might suffice to be able to generate random processes with (*i*) different spectra and exactly the same pdf, or (*ii*) different pdf's and exactly the same spectrum. The first of these objectives is accomplished by keeping the nonlinearity fixed and varying the spectrum of the shaping filter. Inspection of eqs. (12) and (13) shows how the second objective may be accomplished.[7] Since the acf in eq. (13) depends only upon the squares of the expansion coefficients, it is evident that two nonlinearities for which one or more coefficients differ in sign (but not in magnitude) will yield identical spectra. The pdf will in general be quite different in the two cases.

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APPENDIX

Derivation of R(τ)

In this appendix we will derive the three equations (14a), (14b), and (14c) of the text. All three derivations depend on a well-known property of the bivariate Gaussian pdf $g(u, v, \rho)$, namely,

$$\frac{\partial g}{\partial \rho} = \frac{\partial^2 g}{\partial u \partial v}. \tag{24}$$

This follows trivially from the representation of g as the Fourier transform of its characteristic function, i.e.,

$$g(u, v, \rho) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j(\omega u + \sigma v)} e^{-\frac{1}{2}(\omega^2 + \sigma^2 + 2\rho\omega\sigma)} d\omega d\sigma. \tag{25}$$

Differentiating this expression with respect to ρ and then with respect to u and v verifies eq. (24).

[In passing we note that Mehler's expansion eq. (11) also follows trivially from eq. (25). For if

$$g(u, v, \rho) = \sum_{0}^{\infty} a_n(u, v) \frac{\rho^n}{n!} \tag{26}$$

it follows that

$$\begin{aligned} a_n(u, v) &= \left. \frac{\partial^n g}{\partial \rho^n} \right|_{\rho=0} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (-\omega\sigma)^n e^{j(\omega u + \sigma v)} e^{-\frac{1}{2}(\omega^2+\sigma^2)} d\omega d\sigma \\ &= \frac{\partial^n p(u)}{\partial u^n} \frac{\partial^n p(v)}{\partial v^n}. \end{aligned} \tag{27}$$

Here $p(.)$ is the univariate Gaussian pdf, eq. (3). However, the Hermite polynomials are *defined* by the relation

$$\frac{d^n p(x)}{dx^n} = (-1)^n p(x) \mathrm{He}_n(x), \tag{28}$$

which thus gives Mehler's expansion.]

Let us differentiate eq. (10) of the text with respect to ρ and use eq. (24) to evaluate the right-hand side. Thus,

$$\frac{dR}{d\rho} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u)F(v) \frac{\partial^2 g}{\partial u \partial v} dudv. \tag{29}$$

Integrating eq. (29) by parts and assuming that $g(u, v, \rho)F(u)F(v)$ vanishes when $u, v \rightarrow \pm\infty$, we get

$$\frac{dR}{d\rho} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F'(u)F'(v)g(u, v, \rho) dudv. \tag{30}$$

For the binary case (8c), $F'(u) = 2\delta(u)$. In this case

$$\frac{dR}{d\rho} = 4g(0, 0, \rho) = \frac{2}{\pi\sqrt{1-\rho^2}}. \tag{31}$$

Since $R = 0$ when $\rho = 0$ (because F is antisymmetric) this immediately gives

$$R = \frac{2}{\pi} \sin^{-1} \rho, \tag{32}$$

which is eq. (14c).

For the nonlinearity in the case of the gamma pdf,

$$F'(u) = \frac{2}{\sqrt{3}} |u| \tag{33a}$$

$$F''(u) = \frac{2}{\sqrt{3}} \operatorname{sign}(u). \tag{33b}$$

Using eq. (24) twice we get

$$\frac{dR}{d\rho} = \frac{4}{3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u||v| g(u, v, \rho) du dv \tag{34a}$$

and

$$\frac{d^2R}{d\rho^2} = \frac{4}{3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \operatorname{sign}(u)\operatorname{sign}(v) g(u, v, \rho) du dv.$$

$$= \frac{8}{3\pi} \sin^{-1}\rho, \tag{34b}$$

where the last step follows from the result just derived for the binary case. Now at $\rho = 0$, $R = 0$ and, from eq. (34a) $dR/d\rho = 8/3\pi$. With these initial values eq. (34b) can be integrated easily to give eq. (14b) of the text.

Finally, for the nonlinearity (8a) $F'(u) = 2\sqrt{3}p(u)$ and, therefore,

$$\frac{dR}{d\rho} = 12 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(u)p(v) g(u, v, \rho) du dv$$

$$= \frac{3}{\pi^2} \frac{1}{\sqrt{1 - \rho^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{(2-\rho^2)u^2 + (2-\rho^2)v^2 - 2\rho uv}{2(1-\rho^2)}} du dv. \tag{35}$$

The integrand in eq. (35) is of the same form as $g(u, v, \rho)$ with a somewhat different quadratic form in the exponent. It is a standard integral whose value is $2\pi/\sqrt{\Delta}$, where Δ is the determinant of the quadratic form. Simple algebraic manipulation then gives

$$\frac{dR}{d\rho} = \frac{6}{\pi} \frac{1}{\sqrt{4 - \rho^2}}. \tag{36}$$

Again since $R = 0$ when $\rho = 0$, this gives

$$R = \frac{6}{\pi} \sin^{-1} \frac{\rho}{2}, \tag{37}$$

which is eq. (14a).