Semidefinite Positive Relaxation of the Maximum-Likelihood Criterion Applied to Multiuser Detection in a CDMA Context

Moussa Abdi, Hassan El Nahas, Alexandre Jard, and Eric Moulines, *Member, IEEE*

***Abstract*—Many signal processing applications reduce to solving combinatorial optimization problems. Recently, semidefinite programming (SDP) has been shown to be a very promising approach to combinatorial optimization, where SDP serves as a tractable convex relaxation of NP-hard problems. In this paper, we present a nonlinear programming algorithm for solving SDP, based on a change of variables that replaces the symmetrical, positive semidefinite variable $\mathbf{X}$ in SDP with a rectangular variable $\mathbf{R}$ according to $\mathbf{X} = \mathbf{R}\mathbf{R}^T$. Very encouraging results are obtained to solve even large-scale combinatorial optimization programs, as the one arising in multiuser detection for code division multiple access (CDMA) systems.**

***Index Terms*—Code division multiple access, low-rank factorization, multiuser detection, nonlinear programming, semidefinite programming.**

I. Maximum-Likelihood Criterion and Its Relaxation

We consider U users transmitting simultaneously in a code division multiple access (CDMA) system a block of N bits. The signal at the receiver (after chip-matched filtering and sampling—see [7]) may be written as $\mathbf{Y} = \mathbf{A}\mathbf{b} + \boldsymbol{\epsilon}$ where the columns of $\mathbf{A}$ are the convolutions of the codes with the channel impulse responses; the vector $\mathbf{b}$ contains the transmitted bits of the different users (over a slot); and $\boldsymbol{\epsilon}$ is an additive white Gaussian noise $\boldsymbol{\epsilon} \sim \mathcal{N}(0, \sigma^2 \mathbf{I})$. The matrix $\mathbf{A}$ is supposed to be known so that the maximum-likelihood detection scheme, proposed and analyzed in [8], consists in finding (assuming binary phase-shift keying modulation)

$$\arg \min_{\mathbf{b} \in \{-1,1\}^N} \|\mathbf{Y} - \mathbf{A}\mathbf{b}\|^2 = \arg \min_{\mathbf{X}(\mathbf{b})} \mathrm{Tr}(\mathbf{Q}\mathbf{X}(\mathbf{b}))$$

where

$$\mathbf{Q} = \begin{pmatrix} 0 & -\mathbf{Y}^H\mathbf{A} \\ -\mathbf{A}^H\mathbf{Y} & \mathbf{A}^H\mathbf{A} \end{pmatrix}$$

$$\mathbf{X}(\mathbf{b}) = \begin{pmatrix} 1 \\ \mathbf{b} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{b}^H \end{pmatrix}.$$

This problem is of course prototypal of many signal processing applications: similar problems arise in space–time transmission schemes, demodulation over fading channels, etc. This optimization problem is well known to be NP-hard. In the past few years, the topic of semidefinite programming (SDP) has received considerable attention to solve combinatorial optimization: SDP can serve as a tractable convex relaxation of NP-hard problems [2]. The principle of SDP is to embed the combinatorial optimization problem into a much simpler convex optimization problem, as explained below. When $\mathbf{b}$ runs through the vectors with coordinates in $\{-1, 1\}$, $\mathbf{X}(\mathbf{b})$ runs through the set of matrices characterized by

1) for all $i \leq N+1$, $\mathbf{X}(\mathbf{b})_{ii} = 1$;
2) $\mathbf{X}(\mathbf{b})$ has rank one;
3) $\mathbf{X}(\mathbf{b}) \in \mathcal{S}_+^{N+1}$

where $\mathcal{S}_+^{N+1}$ is the positive cone of nonnegative symmetrical matrices of dimension $N+1$. Our initial problem is then equivalent to $\min_{\mathbf{X}} \mathrm{Tr}(\mathbf{Q}\mathbf{X})$ under the constraints $\mathbf{X} \in \mathcal{S}_+^{N+1}$, $\mathrm{rank}(\mathbf{X}) = 1$, and $\mathbf{X}_{ii} = 1$ for all $i \leq N+1$. Ignoring the nonconvex “rank one” constraint, we are left with a convex optimization problem:

$$(\mathrm{SDP}) \begin{cases} \min_{\mathbf{X}} \mathrm{Tr}(\mathbf{Q}\mathbf{X}) \\ \mathbf{X} \in \mathcal{S}_+^{N+1}, \\ \mathbf{X}_{ii} = 1, \text{ for all } i \leq N+1. \end{cases} \tag{1}$$

This SDP optimization problem is linear with linear constraints and can be solved in polynomial time using interior-point methods, as suggested in [4]. However, these methods are still quite time- and memory-intensive and are not adapted for our communication problem (see below).

To retrieve the optimal sequence, a simple solution consists in computing the eigenvector

$$\mathbf{z} = \begin{pmatrix} z_0 \\ \mathbf{z}_1 \end{pmatrix}$$

associated to the greatest eigenvalue of the optimal solution $\mathbf{X}^*$, where $z_0 \geq 0$ is its first coordinate, and then setting $\hat{\mathbf{b}}^* = \mathrm{sgn}(\mathbf{z}_1)$.

II. Numerical Solution of the SDP Program

There are many papers in the literature proposing solutions to SDP (e.g., see [2], [4]). Interior-point methods, though polynomial in time, are not applicable to the current problem, due to the typical size N which is from hundreds to thousands in the CDMA context considered herein.

Manuscript received July 30, 2001; revised January 10, 2002. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. Gaetano Scarano.

M. Abdi, H. E. Nahas, and A. Jard are with the Nortel Networks, 78928 Yvelines Cedex 9, France (e-mail: mabdi@nortelnetworks.com; helhassa@nortelnetworks.com; agard@nortelnetworks.com).

E. Moulines is with the Ecole Nationale Supérieure des Télécommunications, Departement Traitement du Signal et des Images, 75013 Paris, France (e-mail: moulines@tsi.enst.fr).

Publisher Item Identifier 10.1109/LSP.2002.800508.

Generally, the constraint $\mathbf{X} \succeq 0$ is the most challenging aspect of solving (1), since the objective function and constraints are only linear in $\mathbf{X}$. Hoping simply to circumvent this difficult constraint, we introduce, after [2], the change of variable $\mathbf{X} = \mathbf{R}\mathbf{R}^T$, where $\mathbf{R} = [\mathbf{R}_1, \ldots, \mathbf{R}_{N+1}]^T$ is a real $(N+1) \times (N+1)$ matrix, which is taken here lower triangular with positive diagonal elements. In terms of the new variables $\mathbf{R}$, the resulting nonlinear program

$$\begin{aligned} \min \{\mathbf{R} &= [\mathbf{R}_1, \ldots, \mathbf{R}_{N+1}]^T, \\ \mathbf{R}_i &\in \mathbb{R}^{N+1}, \mathrm{Tr}(\mathbf{Q}\mathbf{R}\mathbf{R}^T), \\ \|\mathbf{R}_i\|^2 &= 1, i = 1, \ldots, (N+1)\} \end{aligned} \tag{2}$$

is easily seen to be equivalent to (1). Note, however, that the objective function and the constraints are no longer linear, but instead quadratic and in general nonconvex. In the applications considered, the number of variables in the transformed problem is $(N+1)(N+2)/2$, which can be prohibitively high. However, this number of variables can be drastically reduced by appealing to a result proven independently in [1] and [6], stating that, under weak additional conditions which may be shown to hold for the problem at hand, an optimal solution of the original SDP problem (1) with rank r satisfying the inequality $r(r+1)/2 \le (N+1)$ exists. Note that a nonnegative symmetrical matrix with rank r can be factored as $\mathbf{X} = \mathbf{R}\mathbf{R}^T$, with $\mathbf{R}$ $((N+1) \times r)$ real rectangular matrix with positive diagonal elements. We can thus use the above result and solve a program similar to (2):

$$\begin{aligned} \min \{\mathbf{R} &= [\mathbf{R}_1, \ldots, \mathbf{R}_{N+1}]^T, \\ \mathbf{R}_i &\in \mathbb{R}^{r}, \mathrm{Tr}(\mathbf{Q}\mathbf{R}\mathbf{R}^T), \\ \|\mathbf{R}_i\|^2 &= 1, i = 1, \ldots, (N+1)\}. \end{aligned} \tag{3}$$

The set of solutions of this problem is included in the set of solutions of (2). The number of variables now is $(N+1)r$ which is much smaller than $\mathcal{O}((N+1)^2)$. In practice, r can be chosen as small as 2, and much smaller than $\sqrt{2(N+1)}$, but this result is experimental and is not yet supported by theoretical claims.

To solve the reduced-rank nonlinear program, several solutions can be considered. Due to lack of space, we only describe here a relaxation technique, which consists in updating successively each row of the matrix $\mathbf{R} = [\mathbf{R}_1, \ldots, \mathbf{R}_{N+1}]^T$, while letting the other rows constants. Denote $\{\mathbf{e}_i\}_{1 \le i \le N+1}$ the canonical basis of $\mathbb{R}^{N+1}$ and $\mathrm{span}(\mathbf{e}_1, \ldots, \mathbf{e}_i)$ the linear space spanned by $\mathbf{e}_1, \ldots, \mathbf{e}_i$. Denote $\mathbf{R}^{(n)} = [\mathbf{R}_1^{(n)}, \ldots, \mathbf{R}_{N+1}^{(n)}]^T$ the current value of the matrix $\mathbf{R}$ at iteration n. Each row $\mathbf{R}_i^{(n)}$ is updated by solving the following constrained optimization program

$$\begin{aligned} \mathbf{R}_i^{(n+1)} &= \arg\min_{\|\mathbf{R}_i\|=1, \mathbf{R}_i \in \mathrm{span}(\mathbf{e}_1, \ldots, \mathbf{e}_i)} \\ &\quad \phi(\mathbf{R}_1^{(n+1)}, \ldots, \mathbf{R}_{i-1}^{(n+1)}, \mathbf{R}_i, \mathbf{R}_{i+1}^{(n)}, \ldots, \mathbf{R}_{N+1}^{(n)}) \end{aligned} \tag{4}$$

where $\phi(\mathbf{R}_1, \ldots, \mathbf{R}_{N+1}) = \mathrm{Tr}(\mathbf{Q}\mathbf{R}\mathbf{R}^T)$. It is interesting to note that each constrained optimization step in the inner loop of the optimization procedure can be solved in closed form

$$\mathbf{R}_i^{(n+1)} = \frac{-\prod_i \left(\sum_{j<i} \mathbf{Q}_{ij} \mathbf{R}_j^{(n+1)} + \sum_{j>i} \mathbf{Q}_{ij} \mathbf{R}_j^{(n)} \right)}{\left\| \prod_i \left(\sum_{j<i} \mathbf{Q}_{ij} \mathbf{R}_j^{(n+1)} + \sum_{j>i} \mathbf{Q}_{ij} \mathbf{R}_j^{(n)} \right) \right\|}$$

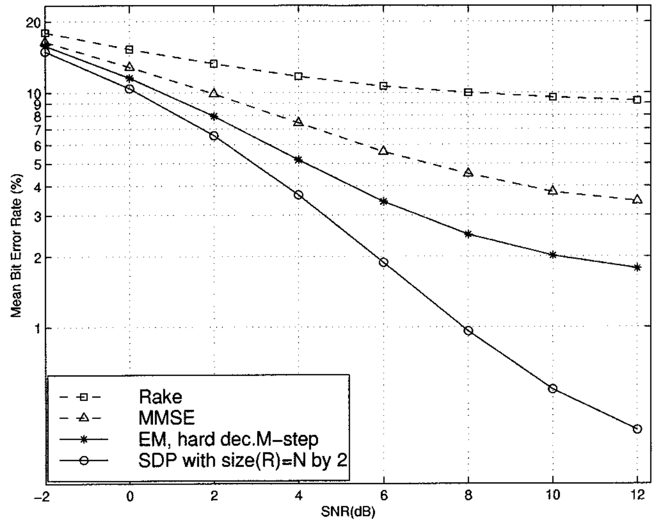

Fig. 1. Comparison of the performance of different multiuser detection algorithms with $U = 4$ users and a spreading factor $S = 4$.

where the operator $\prod_i$ is the projector on $\mathrm{span}(\mathbf{e}_1, \ldots, \mathbf{e}_i)$. Other solutions include standard nonconvex unconstrained optimization techniques and, in particular, the limited-memory Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm, which is not described here due to lack of space.

III. Numerical Results

The simulations were run with a multipaths channel, defined by the European Telecommunications Standard Institute (ETSI) and called "Vehicular B." The length-W impulse response of this channel, which is not stationary, consists in a multipaths Rayleigh fading channel. This impulse response (the fadings as well as the delays) is supposed to be known.

We considered slots of 320 chips here for the purpose of simulations. In the scenario considered here, there are $U = 4$ users with the same spreading factor $S = 4$ sharing the channel. Hence the total number of transmitted bits by slot is $N = N_b U = 320$, which is admittedly a large number for a combinatorial optimization problem, and prohibits the use of interior-point methods in our context. The signal-to-noise ratio (SNR), defined here as the energy per transmitted S chips divided by the Gaussian noise spectral level σ^2, varies from -2 to 12 dB. We simulated four algorithms: the rake receiver, the minimum mean square error (MMSE) receiver, the expectation–maximization (EM) receiver with hard-decision M-step (see [5]), and the rank-two SDP.

Since the correlation matrix $\mathbf{A}^H\mathbf{A}$ is banded with band $\beta = \lceil 1 + (W-1)/S \rceil U$ we need (see [3]) $(N_b U)\beta^2 + 7N_b U\beta + 2N_b U$ flops to solve the linear system corresponding to the MMSE (if we use a band Cholesky procedure), and the EM algorithm can be done in $2N_b U\beta$ flops. On the other hand, if we use a relaxation (SDP) algorithm with $\mathbf{R}$ of size $(N_b U + 1) \times r$ and N_i iterations, we need approximately $2rN_i\beta N_b U$ flops. In our simulation we took $r = 2$ and $N_i = 5$ so that the MMSE is approximately three times as complex as SDP, which is twice as complex as the EM algorithm.

IV. Conclusion

In this letter, we have explored a new relaxation scheme derived from the optimal maximum-likelihood detector, and we obtained a suboptimal and less complex algorithm. We gave an efficient iterative method that converges very quickly. The results of the simulations showed that this algorithm outperforms the classical linear MMSE, and even the EM-based receiver. Moreover, we also gave some variants of our algorithm that have a reduced complexity with almost the same performances.

References

[1] A. Barvinok, "Problems of distance geometry and convex properties of quadratic maps," *Discr. Comput. Geom.*, vol. 13, pp. 189–202, 1995.

[2] S. Burer and R. D. C. Monteiro, "An efficient algorithm for solving the MAXCUT SDP relaxation,", Georgia Tech, Atlanta, 1998.

[3] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 3rd ed. Baltimore, MD: John Hopkins Univ. Press, 1996.

[4] C. Helmberg, F. Rendl, R. J. Vanderbei, and H. Wolkowicz, "An interior point method for Semidefinite Programming," Univ. Waterloo, Waterloo, ON, Canada, Res. Rep.

[5] L. B. Nelson and H. V. Poor, "Iterative multiuser receivers for CDMA channels: An EM-based approach," *IEEE Trans. Commun.*, vol. 44, Dec. 1996.

[6] G. Pataki, "On the rank of extreme matrices in semidefinite programs and the multiplicity of optimal eigenvalues," *Math. Oper. Res.*, vol. 23, pp. 339–358, 1998.

[7] S. Verdú, *Multiuser Detection*. Cambridge, U.K.: Cambridge Univ. Press, 1998.

[8] ——, "Minimum probability of error for asynchronous Gaussian multiple-access channels," *IEEE Trans. Inform. Theory*, vol. IT-32, pp. 85–96, Jan. 1986.