Real-Time Embedded Convex Optimization

Stephen Boyd

joint work with Michael Grant, Jacob Mattingley, Yang Wang

Electrical Engineering Department, Stanford University

Spencer Schantz Lecture, Lehigh University, 12/6/10

Outline

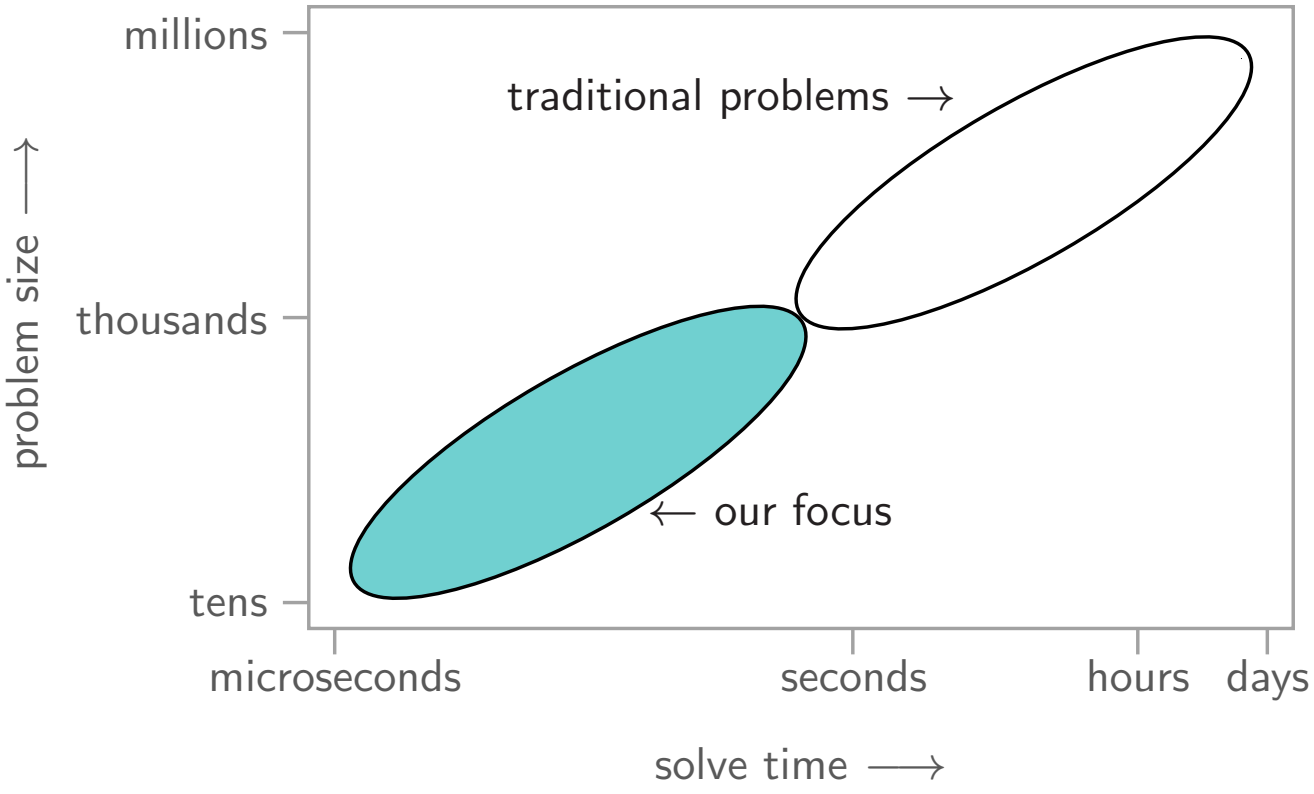
- Real-time embedded convex optimization
- Examples
- Parser/solvers for convex optimization
- Code generation for real-time embedded convex optimization

Embedded optimization

- embed solvers in real-time applications
- $i.e.$, **solve an optimization problem at each time step**
- used now for applications with hour/minute time-scales
  - process control
  - supply chain and revenue management
  - trading

What's new

embedded optimization at **millisecond/microsecond time-scales**



Applications

- real-time resource allocation
  - update allocation as objective, resource availabilities change
- signal processing
  - estimate signal by solving optimization problem over sliding window
  - replace least-squares estimates with robust (Huber, ℓ_1) versions
  - re-design (adapt) coefficients as signal/system model changes
- control
  - closed-loop control via rolling horizon optimization
  - real-time trajectory planning
- all of these done now, on long (minutes or more) time scales
  but could be done on millisecond/microsecond time scales

Outline

- Real-time embedded convex optimization
- Examples
- Parser/solvers for convex optimization
- Code generation for real-time embedded convex optimization

Multi-period processor speed scheduling

- processor adjusts its speed $s_t \in [s^{\min}, s^{\max}]$ in each of T time periods
- energy consumed in period t is $\phi(s_t)$; total energy is $E = \sum_{t=1}^{T} \phi(s_t)$
- n jobs
  - job i available at $t = A_i$; must finish by deadline $t = D_i$
  - job i requires total work $W_i \geq 0$
- $S_{ti} \geq 0$ is effective processor speed allocated to job i in period t

$$s_t = \sum_{i=1}^{n} S_{ti}, \qquad \sum_{t=A_i}^{D_i} S_{ti} \geq W_i$$

Minimum energy processor speed scheduling

- choose speeds s_t and allocations S_{ti} to minimize total energy E

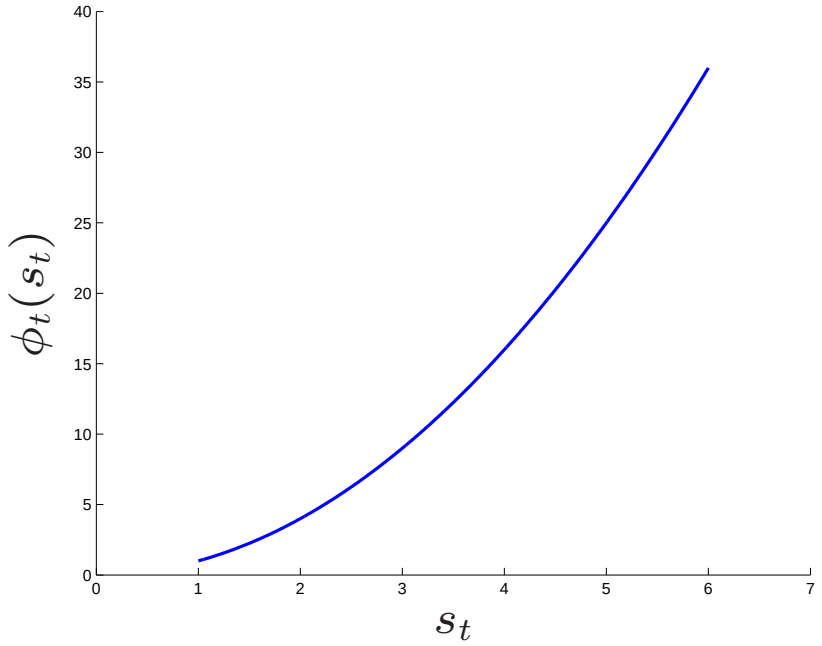
$$\begin{array}{ll} \text{minimize} & E = \sum_{t=1}^{T} \phi(s_t) \\ \text{subject to} & s^{\min} \leq s_t \leq s^{\max}, \quad t = 1, \ldots, T \\ & s_t = \sum_{i=1}^{n} S_{ti}, \quad t = 1, \ldots, T \\ & \sum_{t=A_i}^{D_i} S_{ti} \geq W_i, \quad i = 1, \ldots, n \end{array}$$

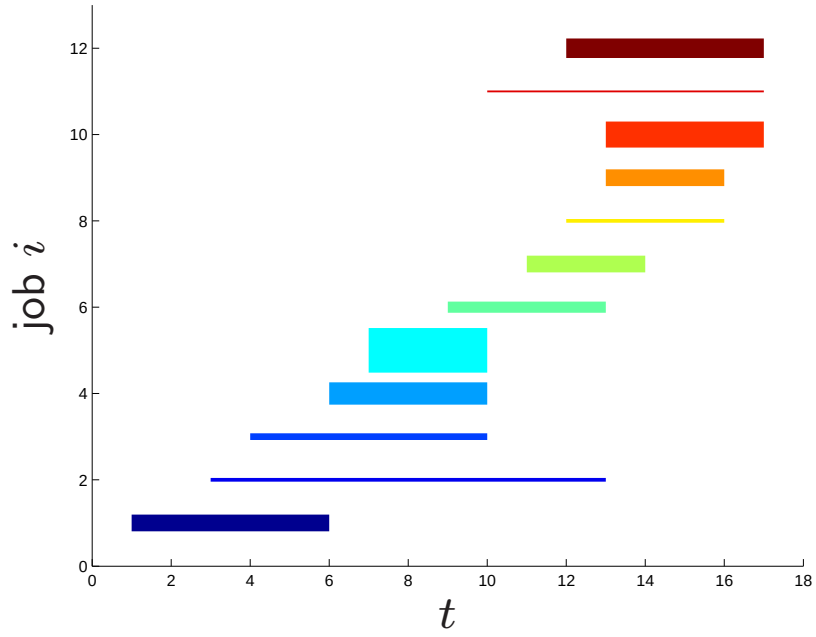
- a convex problem when ϕ is convex

- more sophisticated versions include
  - multiple processors
  - other constraints (thermal, speed slew rate, . . .)
  - stochastic models for (future) data

Example

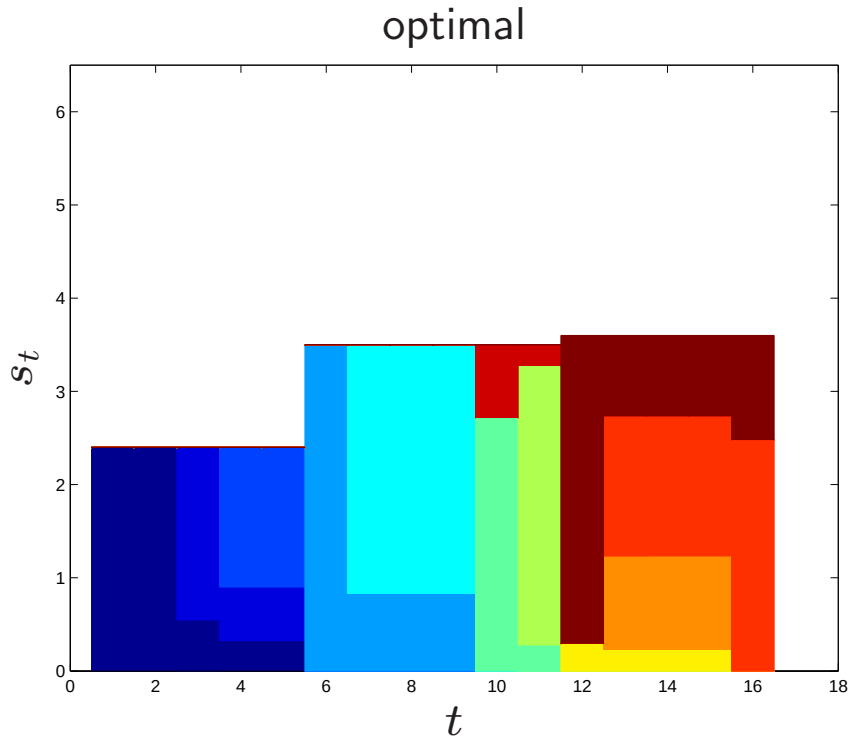
- $T = 16$ periods, $n = 12$ jobs
- $s^{\min} = 1$, $s^{\max} = 6$, $\phi(s_t) = s_t^2$
- jobs shown as bars over $[A_i, D_i]$ with area $\propto W_i$

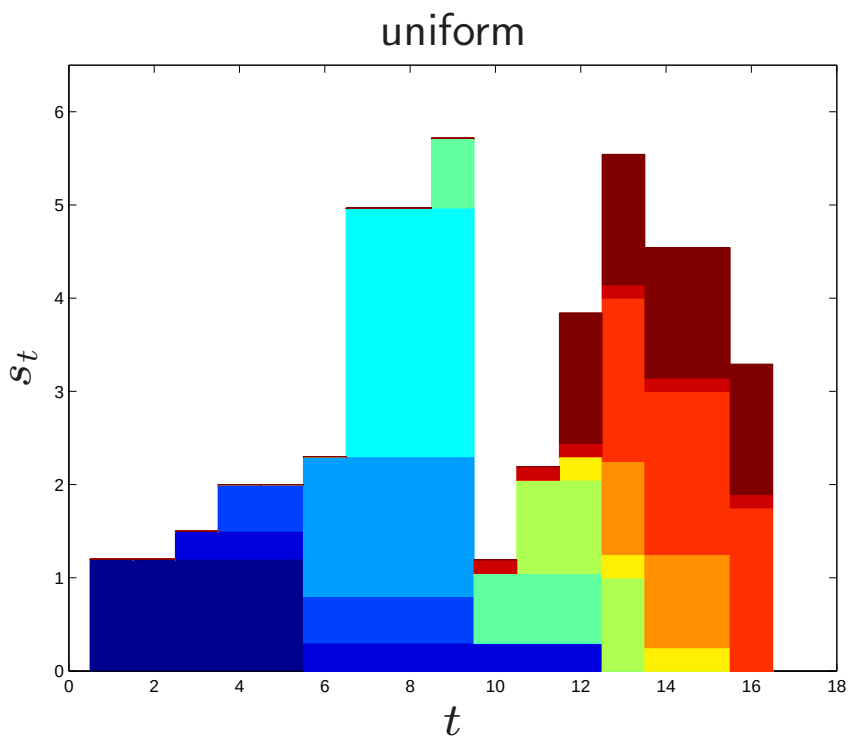




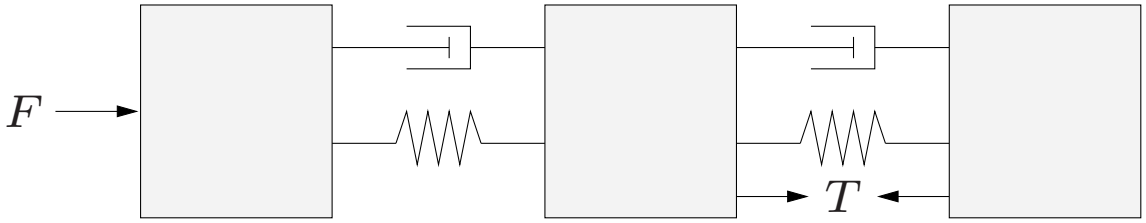
Optimal and uniform schedules

- uniform schedule: $S_{ti} = W_i/(D_i - A_i)$; gives $E^{\mathrm{unif}} = 374.1$
- optimal schedule $S^\star_{ti}$ gives $E^\star = 167.1$





Minimum time control with active vibration supression

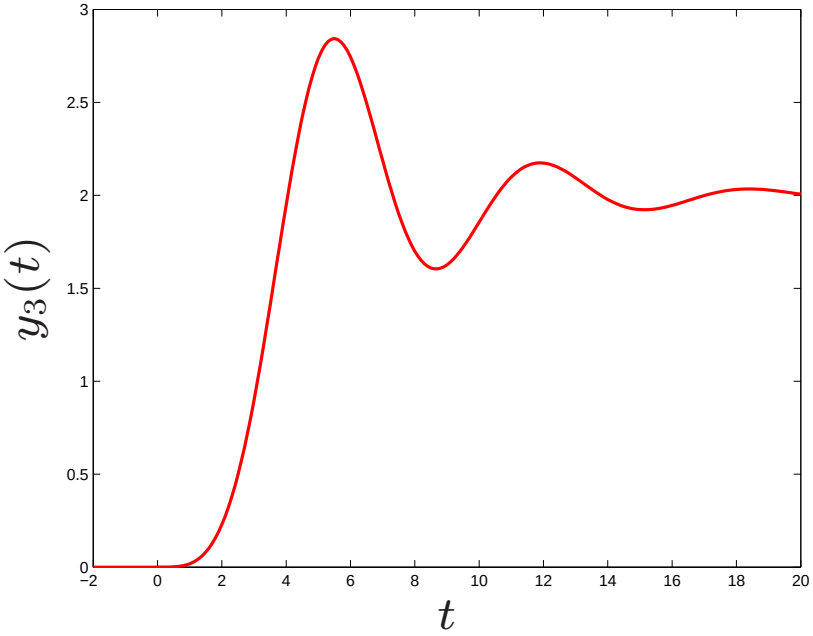


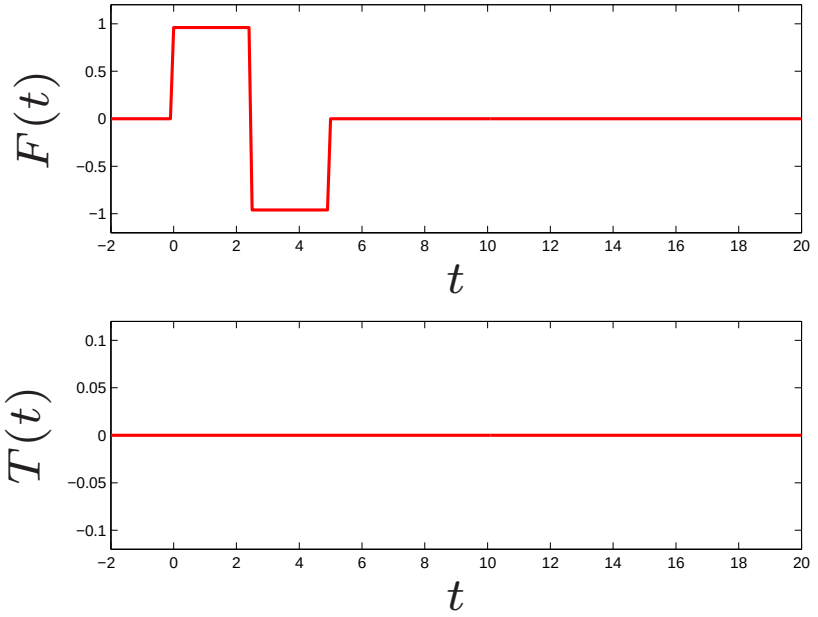
- force $F(t)$ moves object modeled as 3 masses (2 vibration modes)
- tension $T(t)$ used for active vibration supression
- goal: move object to commanded position as quickly as possible, with

$$|F(t)| \leq 1, \qquad |T(t)| \leq 0.1$$

Ignoring vibration modes

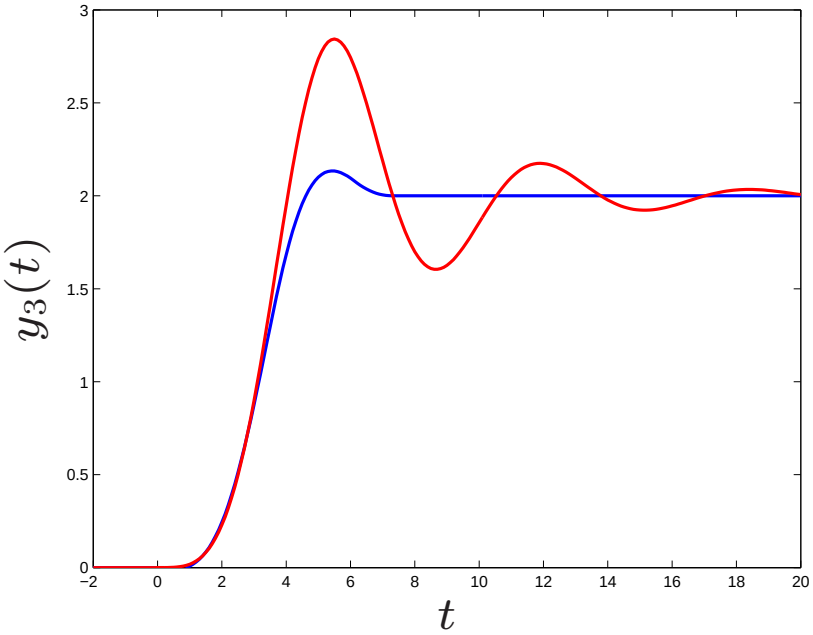
- treat object as single mass; apply only F
- analytical ('bang-bang') solution

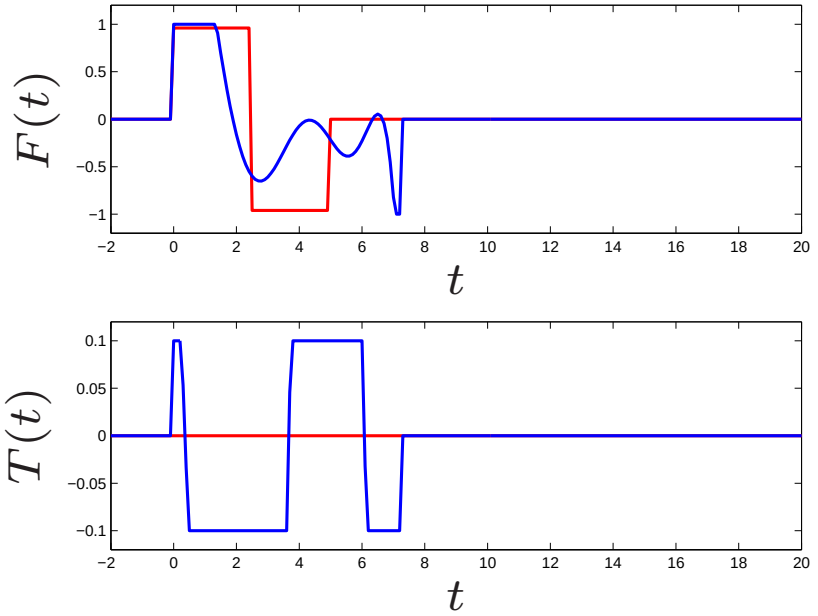




With vibration modes

- no analytical solution, but reduces to a quasiconvex problem
- can be solved by solving a small number of convex problems





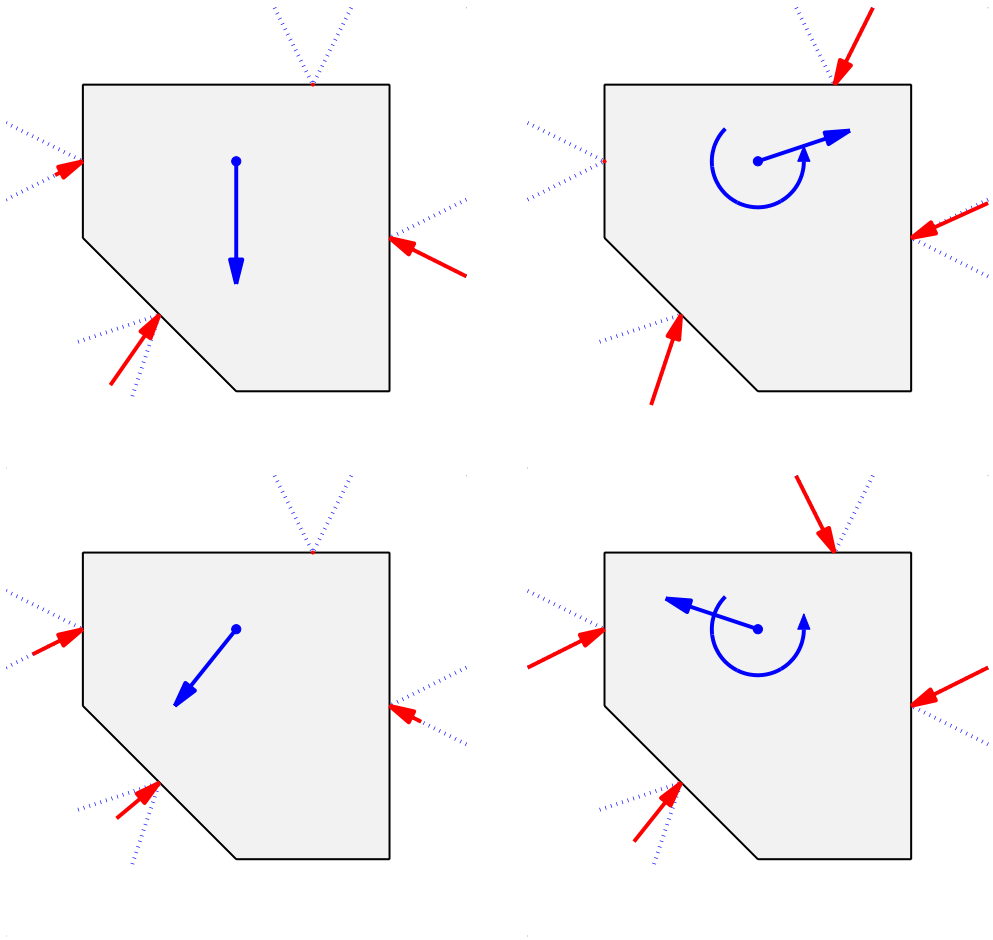
Grasp force optimization

- choose K grasping forces on object to
  - resist external wrench (force and torque)
  - respect friction cone constraints
  - minimize maximum grasp force
- convex problem (second-order cone program or SOCP):

$$
\begin{array}{lll}
\text{minimize} & \max_i \|f^{(i)}\|_2 & \textit{max contact force} \\
\text{subject to} & \sum_i Q^{(i)} f^{(i)} = f^{\text{ext}} & \textit{force equillibrium} \\
& \sum_i p^{(i)} \times (Q^{(i)} f^{(i)}) = \tau^{\text{ext}} & \textit{torque equillibrium} \\
& \mu_i f_z^{(i)} \geq \left(f_x^{(i)2} + f_y^{(i)2} \right)^{1/2} & \textit{friction cone constraints}
\end{array}
$$

variables $f^{(i)} \in \mathbf{R}^3$, $i = 1, \ldots, K$ (contact forces)

Example



Grasp force optimization solve times

- example with $K = 5$ fingers (grasp points)
- reduces to SOCP with 15 vars, 6 eqs, 5 3-dim SOCs
- custom code solve time: 50μs (SDPT3: 100ms)

Robust Kalman filtering

- estimate state of a linear dynamical system driven by IID noise
- sensor measurements have occasional outliers (failures, jamming, . . .)
- model: $x_{t+1} = Ax_t + w_t, \quad y_t = Cx_t + v_t + z_t$
  - $w_t \sim \mathcal{N}(0, W)$, $v_t \sim \mathcal{N}(0, V)$
  - z_t is **sparse**; represents outliers, failures, . . .
- (steady-state) Kalman filter (for case $z_t = 0$):
  - time update: $\hat{x}_{t+1|t} = A\hat{x}_{t|t}$
  - measurement update: $\hat{x}_{t|t} = \hat{x}_{t|t-1} + L(y_t - C\hat{x}_{t|t-1})$
- we'll replace measurement update with robust version to handle outliers

Measurement update via optimization

- standard KF: $\hat{x}_{t|t}$ is solution of quadratic problem

$$\begin{array}{ll} \text{minimize} & v^T V^{-1} v + (x - \hat{x}_{t|t-1})^T \Sigma^{-1} (x - \hat{x}_{t|t-1}) \\ \text{subject to} & y_t = Cx + v \end{array}$$

  with variables x, v (simple analytic solution)

- robust KF: choose $\hat{x}_{t|t}$ as solution of convex problem

$$\begin{array}{ll} \text{minimize} & v^T V^{-1} v + (x - \hat{x}_{t|t-1})^T \Sigma^{-1} (x - \hat{x}_{t|t-1}) + \lambda \|z\|_1 \\ \text{subject to} & y_t = Cx + v + z \end{array}$$

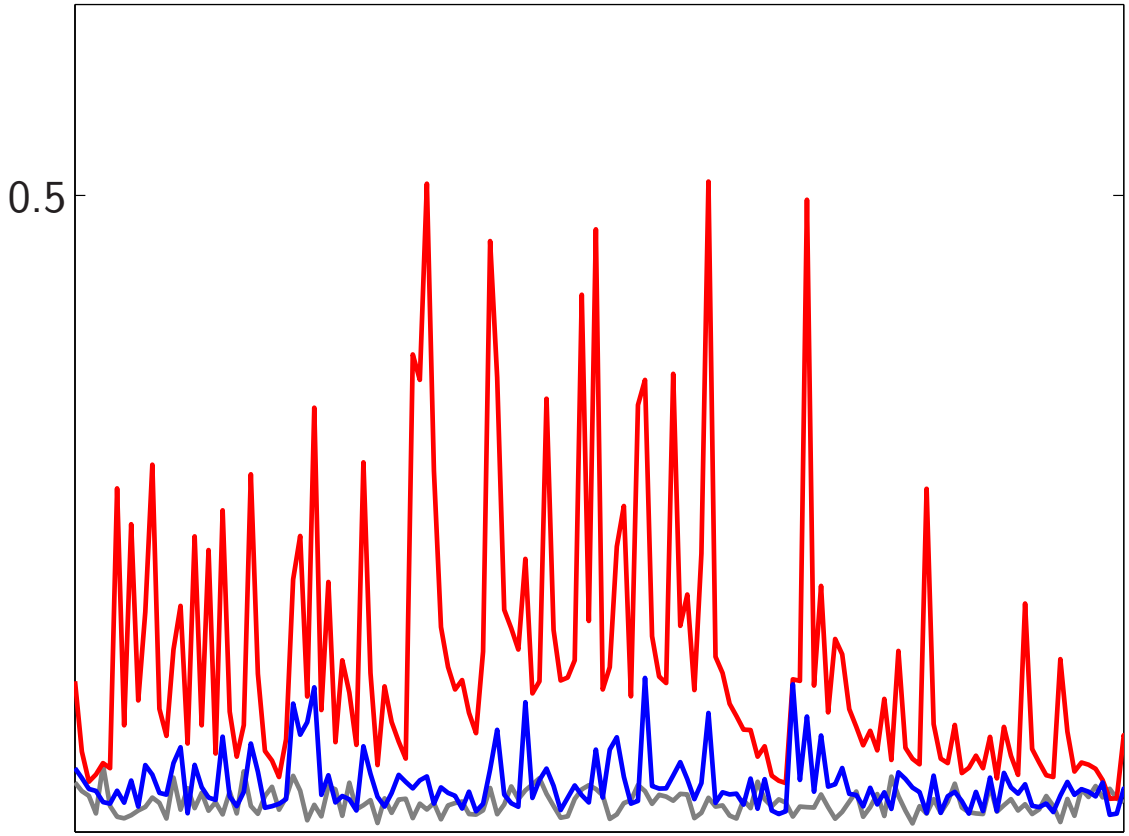
with variables x, v, z (requires solving a QP)

Example

- 50 states, 15 measurements
- with prob. 5%, measurement components replaced with $(y_t)_i = (v_t)_i$
- so, get a flawed measurement ($i.e.$, $z_t \neq 0$) every other step (or so)

State estimation error

$\|x - \hat{x}_{t|t}\|_2$ for KF (red); robust KF (blue); KF with $z = 0$ (gray)

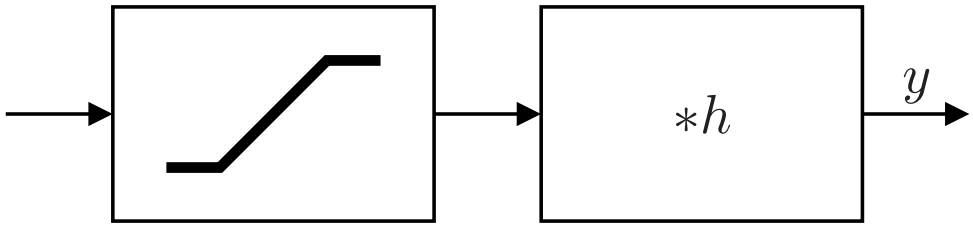


Robust Kalman filter solve time

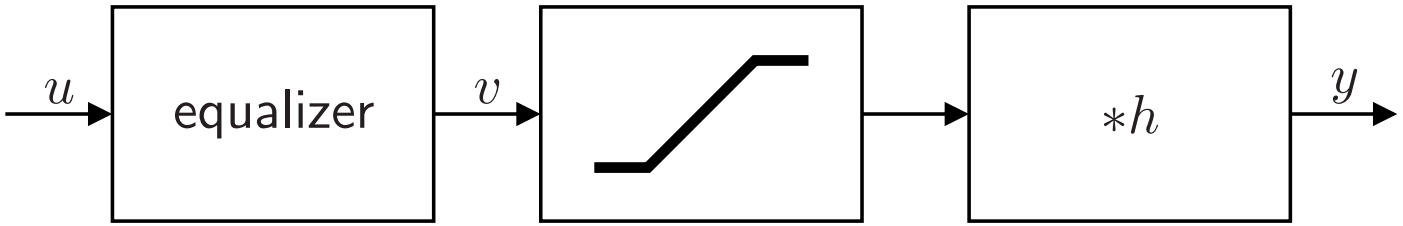
- robust KF requires solution of QP with 95 vars, 15 eqs, 30 ineqs
- automatically generated code solves QP in 120 μs (SDPT3: 120 ms)
- standard Kalman filter update requires 10 μs

Linearizing pre-equalizer

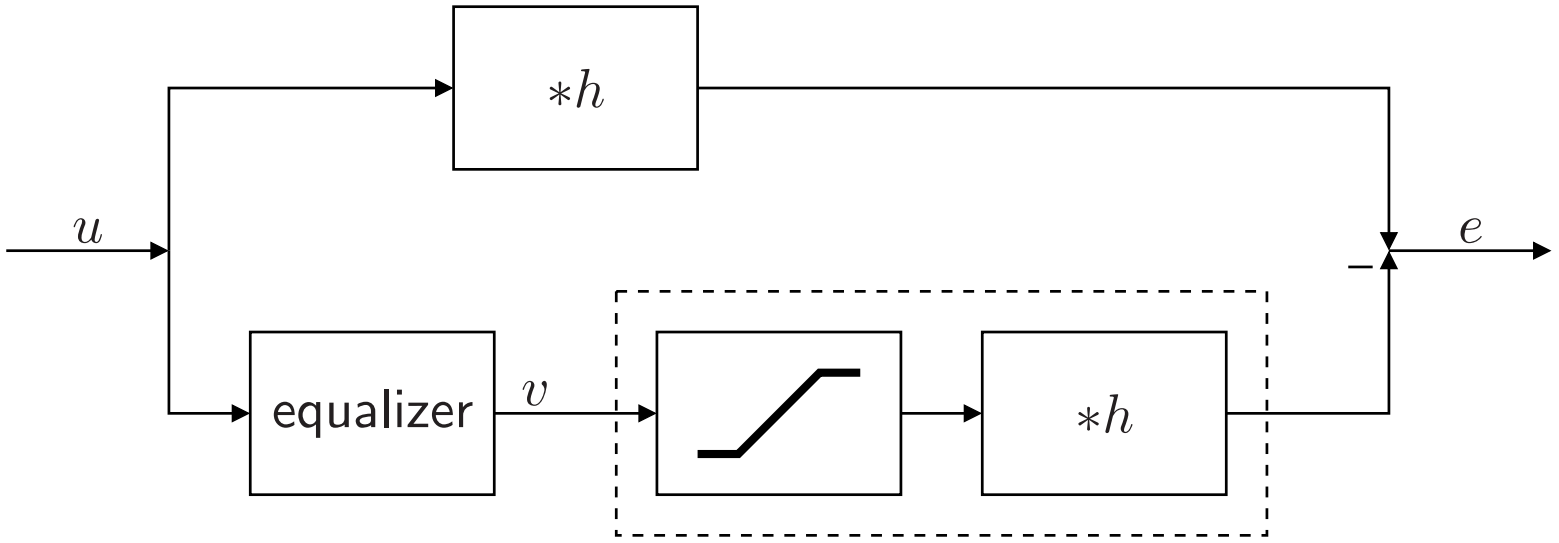
- linear dynamical system with input saturation



- we'll design pre-equalizer to compensate for saturation effects



Linearizing pre-equalizer



- goal: minimize error e (say, in mean-square sense)
- pre-equalizer has T sample look-ahead capability

- system: $x_{t+1} = Ax_t + B\,\mathbf{sat}(v_t), \quad y_t = Cx_t$
- (linear) reference system: $x^{\rm ref}_{t+1} = Ax^{\rm ref}_t + Bu_t, \quad y^{\rm ref}_t = Cx^{\rm ref}_t$
- $e_t = Cx^{\rm ref}_t - Cx_t$
- state error $\tilde{x}_t = x^{\rm ref}_t - x_t$ satisfies

$$\tilde{x}_{t+1} = A\tilde{x}_t + B(u_t - v_t), \quad e_t = C\tilde{x}_t$$

- to choose v_t, solve QP

$$\begin{array}{ll} \text{minimize} & \sum_{\tau=t}^{t+T} e_\tau^2 + \tilde{x}_{t+T+1}^T P \tilde{x}_{t+T+1} \\ \text{subject to} & \tilde{x}_{\tau+1} = A\tilde{x}_\tau + B(u_\tau - v_\tau), \quad e_\tau = C\tilde{x}_\tau, \quad \tau = t, \ldots, t+T \\ & |v_\tau| \leq 1, \quad \tau = t, \ldots, t+T \end{array}$$

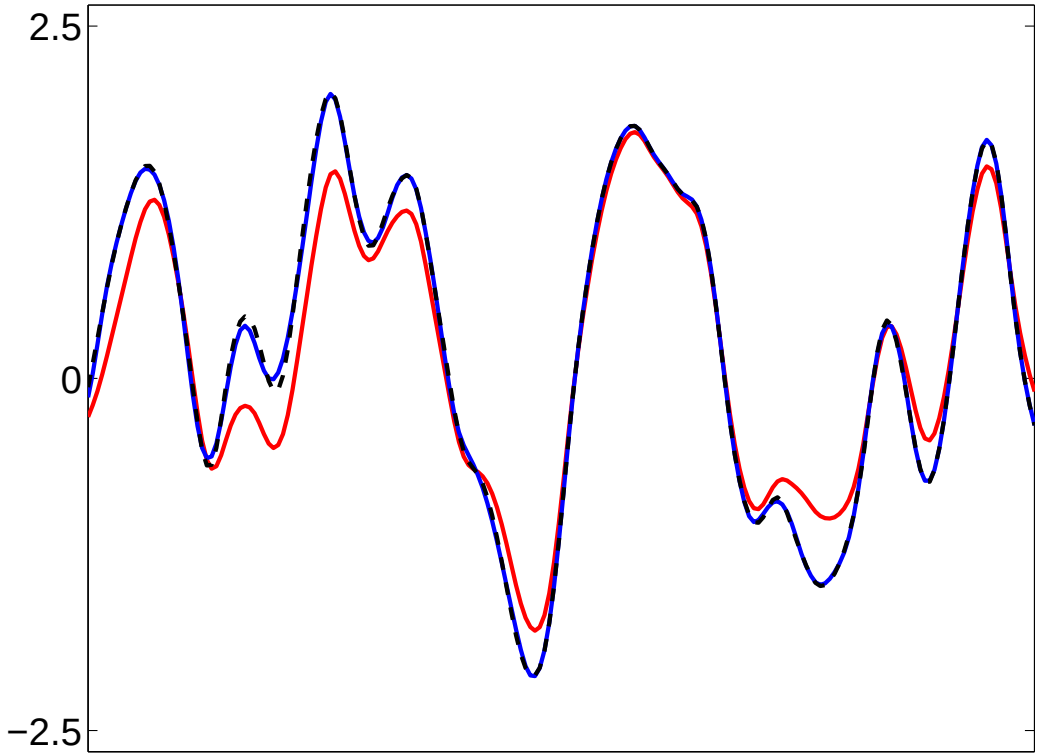
P gives final cost; obvious choice is output Grammian

Example

- state dimension $n = 3$; h decays in around 35 samples
- pre-equalizer look-ahead $T = 15$ samples
- input u random, saturates ($|u_t| > 1$) 20% of time

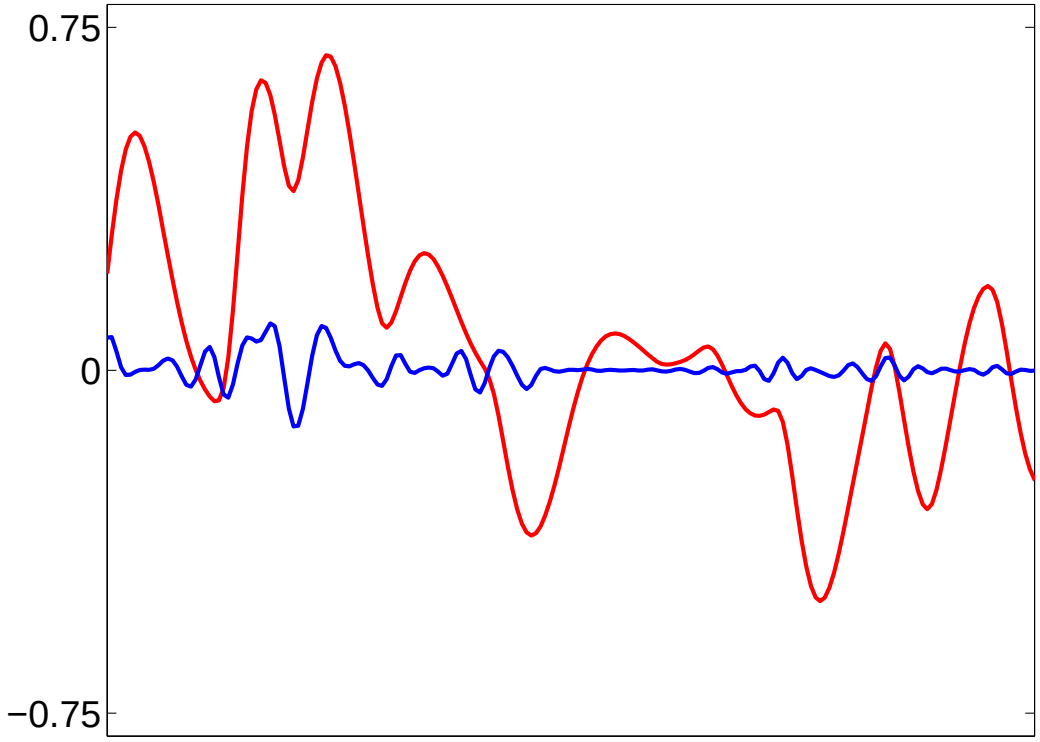
Outputs

desired (black), no compensation (red), equalized (blue)



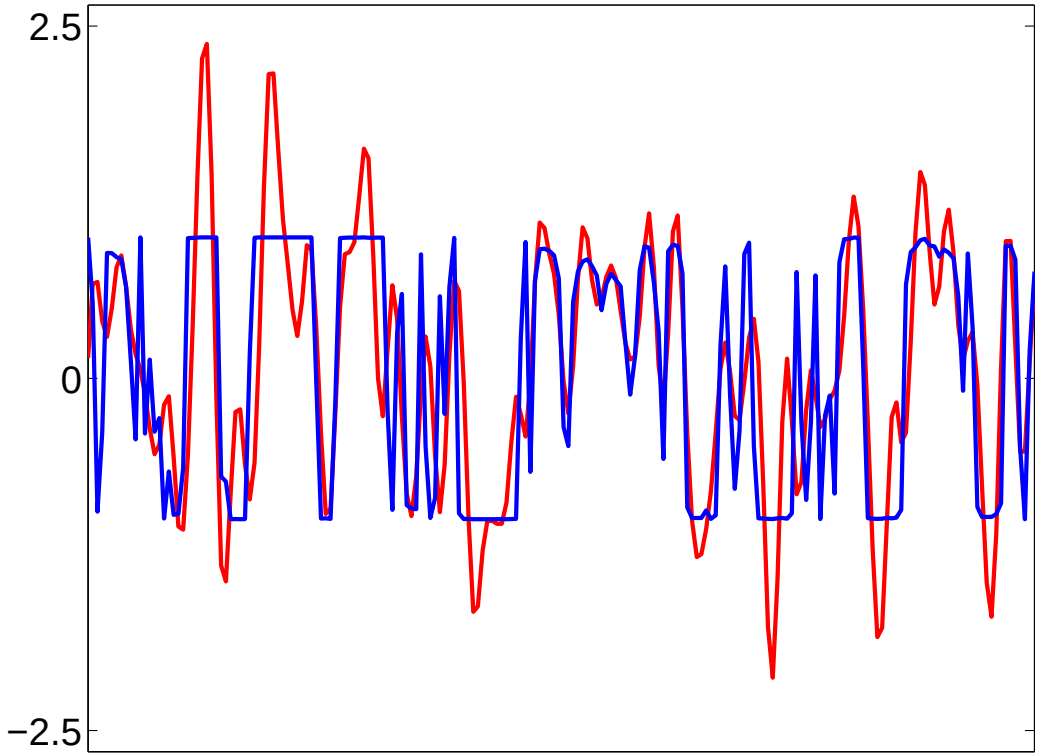
Errors

no compensation (red), with equalization (blue)



Inputs

no compensation (red), with equalization (blue)



Linearizing pre-equalizer solve time

- pre-equalizer problem reduces to QP with 96 vars, 63 eqs, 48 ineqs
- automatically generated code solves QP in 600μs (SDPT3: 310ms)

Constrained linear quadratic stochastic control

- linear dynamical system: $x_{t+1} = Ax_t + Bu_t + w_t$
  - $x_t \in \mathbf{R}^n$ is state; $u_t \in \mathcal{U} \subset \mathbf{R}^m$ is control input
  - w_t is IID zero mean disturbance
- $u_t = \phi(x_t)$, where $\phi : \mathbf{R}^n \to \mathcal{U}$ is (state feedback) policy
- objective: minimize average expected stage cost ($Q \geq 0$, $R > 0$)

$$J = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbf{E}\left(x_t^T Q x_t + u_t^T R u_t\right)$$

- constrained LQ stochastic control problem: choose ϕ to minimize J

Constrained linear quadratic stochastic control

- optimal policy has form

$$\phi(z) = \operatorname*{argmin}_{v \in \mathcal{U}} \{v^T R v + \mathbf{E}\, V(Az + Bv + w_t)\}$$

  where V is Bellman function

  - but V is hard to find/describe except when $\mathcal{U} = \mathbf{R}^m$ (in which case V is quadratic)

- many heuristic methods give suboptimal policies, *e.g.*
  - projected linear control
  - control-Lyapunov policy
  - model predictive control, certainty-equivalent planning

Control-Lyapunov policy

- also called approximate dynamic programming, horizon-1 model predictive control
- CLF policy is

$$\phi_{\rm clf}(z) = \operatorname*{argmin}_{v \in \mathcal{U}} \{ v^T R v + \mathbf{E}\, V_{\rm clf}(Az + Bv + w_t) \}$$

  where $V_{\rm clf} : \mathbf{R}^n \to \mathbf{R}$ is the control-Lyapunov function
- evaluating $u_t = \phi_{\rm clf}(x_t)$ requires solving an optimization problem at **each step**
- many tractable methods can be used to find a good $V_{\rm clf}$
- often works really well

Quadratic control-Lyapunov policy

- assume
  - polyhedral constraint set: $\mathcal{U} = \{v \mid Fv \leq g\}$, $g \in \mathbf{R}^k$
  - quadratic control-Lyapunov function: $V_{\mathrm{clf}}(z) = z^T P z$
- evaluating $u_t = \phi_{\mathrm{clf}}(x_t)$ reduces to solving QP

$$\begin{array}{ll} \text{minimize} & v^T R v + (Az + Bv)^T P (Az + Bv) \\ \text{subject to} & Fv \leq g \end{array}$$

Control-Lyapunov policy evaluation times

- t_{clf}: time to evaluate $\phi_{\mathrm{clf}}(z)$
- t_{lin}: linear policy $\phi_{\mathrm{lin}}(z) = Kz$
- t_{kf}: Kalman filter update
- (SDPT3 times around $1000\times$ larger)

n	m	k	t_{clf} (μs)	t_{lin} (μs)	t_{kf} (μs)
15	5	10	35	1	1
50	15	30	85	3	9
100	10	20	67	4	40
1000	30	60	298	130	8300

Outline

- Real-time embedded convex optimization
- Examples
- Parser/solvers for convex optimization
- Code generation for real-time embedded convex optimization

Parser/solvers for convex optimization

- specify convex problem in natural form
  - declare optimization variables
  - form convex objective and constraints using a specific set of atoms and calculus rules (**disciplined convex programming**)
- problem is convex-by-construction
- easy to parse, automatically transform to standard form, solve, and transform back
- implemented using object-oriented methods and/or compiler-compilers
- huge gain in productivity (rapid prototyping, teaching, research ideas)

Example: cvx

- parser/solver written in Matlab
- convex problem, with variable $x \in \mathbf{R}^n$; A, b, λ, F, g constants

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2 + \lambda\|x\|_1 \\ \text{subject to} & Fx \leq g \end{array}$$

- cvx specification:

```
cvx_begin
    variable x(n)      % declare vector variable
    minimize (norm(A*x-b,2) + lambda*norm(x,1))
    subject to  F*x <= g
cvx_end
```

when `cvx` processes this specification, it

- verifies convexity of problem
- generates equivalent cone problem (here, an SOCP)
- solves it using `SDPT3` or `SeDuMi`
- transforms solution back to original problem

the `cvx` code is easy to read, understand, modify

The same example, transformed by 'hand'

transform problem to SOCP, call `SeDuMi`, reconstruct solution:

```
% Set up big matrices.
[m,n] = size(A); [p,n] = size(F);
AA = [speye(n), -speye(n), speye(n), sparse(n,p+m+1); ...
      F, sparse(p,2*n), speye(p), sparse(p,m+1); ...
      A, sparse(m,2*n+p), speye(m), sparse(m,1)];
bb = [zeros(n,1); g; b];
cc = [zeros(n,1); gamma*ones(2*n,1); zeros(m+p,1); 1];
K.f = m; K.l = 2*n+p; K.q = m + 1;      % specify cone
xx = sedumi(AA, bb, cc, K);             % solve SOCP
x = x(1:n);                             % extract solution
```

Outline

- Real-time embedded convex optimization
- Examples
- Parser/solvers for convex optimization
- Code generation for real-time embedded convex optimization

General vs. embedded solvers

- general solver (say, for QP)
  - handles single problem instances with any dimensions, sparsity pattern
  - typically optimized for large problems
  - must deliver high accuracy
  - variable execution time: stops when tolerance achieved
- embedded solver
  - solves many instances of the same problem family (dimensions, sparsity pattern) with different data
  - solves small or smallish problems
  - can deliver lower (application dependent) accuracy
  - often must satisfy hard real-time deadline

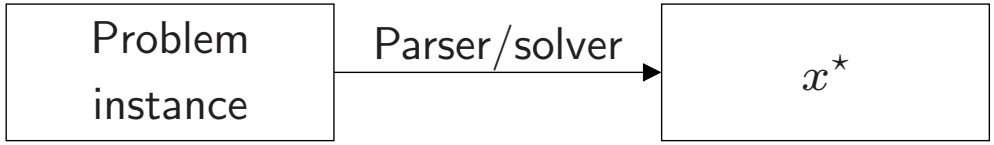
Embedded solvers

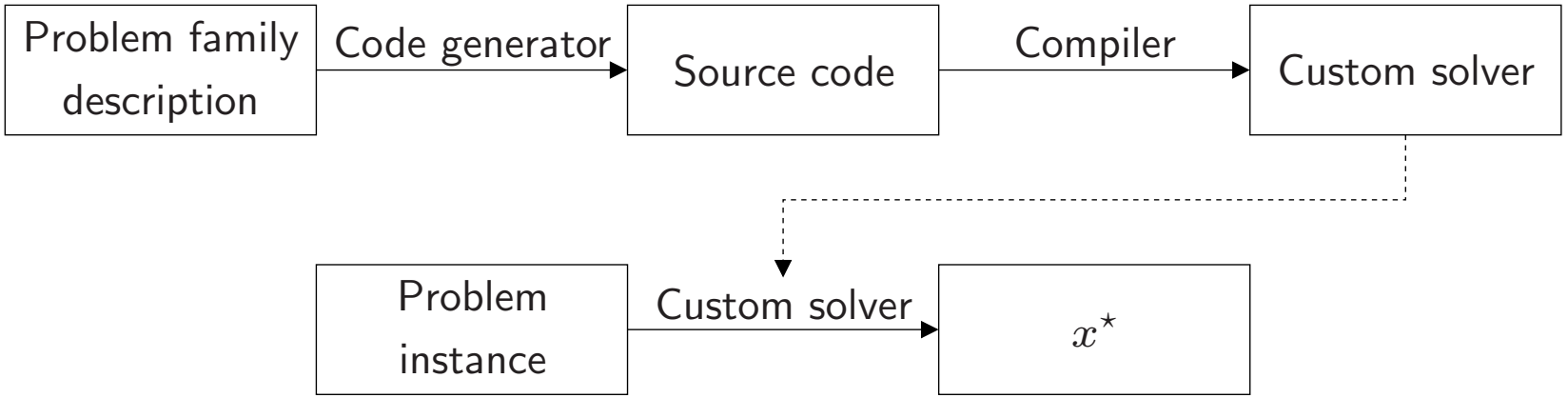
- (if a general solver works, use it)
- otherwise, develop custom code
  - by hand
  - automatically via code generation
- can exploit known sparsity pattern, data ranges, required tolerance at solver code development time
- we've had good results with interior-point methods;
  other methods ($e.g.$, active set, first order) might work well too
- typical speed-up over (efficient) general solver: **100–10000**$\times$

Convex optimization solver generation

- specify convex problem **family** in natural form, via disciplined convex programming
  - declare optimization variables, parameters
  - form convex objective and constraints using a specific set of atoms and calculus rules
- code generator
  - analyzes problem structure (dimensions, sparsity, . . .)
  - chooses elimination ordering
  - generates solver code for specific problem family
- idea:
  - spend (perhaps much) time generating code
  - save (hopefully much) time solving problem instances

Parser/solver vs. code generator





`cvxgen` code generator

- handles problems transformable to QP
- accepts high-level problem family description
- generates flat C source, auxiliary files, documentation
  - no libraries, almost branch-free
- primal-dual interior-point method with iteration limit
  - direct LDL^T factorization of KKT matrix
  - regularization and iterative refinement for extreme reliability
  - (slow) method to determine static variable ordering
  - explicit factorization code generated

cvxgen example specification

```
dimensions
  n = 30  # assets.
parameters
  alpha (n)  # mean returns.
  lambda nonnegative  # risk aversion.
  Sigma (n,n) psd  # covariance.
  eta nonnegative  # limit on total short position.
variables
  w (n)  # asset weights.
maximize
  alpha'*w - lambda*quad(w,Sigma)   # risk adjusted mean return.
subject to
  sum(w) == 1  # budget constraint.
  sum(neg(w)) <= eta   # limit on total short position.
```

Sample solve times for `cvxgen` generated code

	Scheduling	*Battery*	*Suspension*
Variables	279	153	104
Constraints	465	357	165
CVX, Intel i7	4.2 s	1.3 s	2.6 s
CVXGEN, Intel i7	850 μs	360 μs	110 μs
CVXGEN, Atom	7.7 ms	4.0 ms	1.0 ms

Conclusions

- can solve convex problems on **millisecond, microsecond** time scales
  - (using existing algorithms, but not using existing codes)
  - there should be many applications
- parser/solvers make rapid prototyping easy
- new code generation methods yield solvers that
  - are extremely fast, even competitive with 'analytical methods'
  - can be embedded in real-time applications

References

- *Automatic Code Generation for Real-Time Convex Optimization* (Mattingley, Boyd)
- *Real-Time Convex Optimization in Signal Processing* (Mattingley, Boyd)
- *Code Generation for Receding Horizon Control* (Mattingley, Boyd)
- *Fast Evaluation of Quadratic Control-Lyapunov Policy* (Wang, Boyd)
- *Fast Model Predictive Control Using Online Optimization* (Wang, Boyd)
- `cvx` (Grant, Boyd, Ye)
- `cvxgen` (Mattingley, Boyd)