ON CONVEX OPTIMIZATION WITHOUT CONVEX REPRESENTATION

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ABSTRACT. We consider the convex optimization problem $\mathbf{P} : \min_{\mathbf{x}}\{f(\mathbf{x}) : \mathbf{x} \in \mathbf{K}\}$ where f is convex continuously differentiable, and $\mathbf{K} \subset \mathbb{R}^n$ is a compact convex set with representation $\{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geq 0, j = 1, \ldots, m\}$ for some continuously differentiable functions (g_j). We discuss the case where the g_j's are not all concave (in contrast with convex programming where they all are). In particular, even if the g_j are not concave, we consider the log-barrier function ϕ_μ with parameter μ, associated with $\mathbf{P}$, usually defined for concave functions (g_j). We then show that any limit point of any sequence $(\mathbf{x}_\mu) \subset \mathbf{K}$ of stationary points of ϕ_μ, $\mu \to 0$, is a Karush-Kuhn-Tucker point of problem $\mathbf{P}$ and a global minimizer of f on $\mathbf{K}$.

1. Introduction

Consider the optimization problem

$$\mathbf{P} : \quad f^* := \min_{\mathbf{x}} \{ f(\mathbf{x}) : \mathbf{x} \in \mathbf{K}\}. \tag{1.1}$$

for some convex and continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, and where the feasible set $\mathbf{K} \subset \mathbb{R}^n$ is defined by:

$$\mathbf{K} := \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geq 0, \quad j = 1, \ldots, m\}, \tag{1.2}$$

for some continuously differentiable functions $g_j : \mathbb{R}^n \to \mathbb{R}$. We say that (g_j), $j = 1, \ldots, m$, is a *representation* of $\mathbf{K}$. When $\mathbf{K}$ is convex and the (g_j) are concave we say that $\mathbf{K}$ has a convex representation.

In the literature, when $\mathbf{K}$ is convex $\mathbf{P}$ is referred to as a convex optimization problem and in particular, every local minimum of f is a global minimum. However if on the one hand *convex optimization* usually refers to minimizing a convex function on a convex set $\mathbf{K}$ without precising its representation (g_j) (see e.g. Ben-Tal and Nemirovsky [1, Definition 5.1.1] or Bertsekas et al. [3, Chapter 2]), on the other hand *convex programming* usually refers to the situation where the representation of $\mathbf{K}$ is also convex, i.e. when all the g_j's are concave. See for instance Ben-Tal and Nemirovsky [1, p. 335], Berkovitz [2, p. 179], Boyd and Vandenberghe [4, p. 7], Bertsekas et al. [3, §3.5.5], Nesterov and Nemirovskii [11, p. 217-218], and Hiriart-Urruty [9]. Convex programming is particularly interesting because under Slater's condition[1], the standard Karush-Kuhn-Tucker (KKT) optimality conditions are not only necessary but also sufficient and in addition, the concavity property of the g_j's is used to prove convergence (and rates of convergence) of specialized algorithms.

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[1]Slater's condition holds if $g_j(\mathbf{x}_0) > 0$ for some $\mathbf{x}_0 \in \mathbf{K}$ and all $j = 1, \ldots, m$.

To the best of our knowledge, little is said in the literature for the specific case where $\mathbf{K}$ is convex but not necessarily its representation, that is, when the functions (g_j) are *not* necessarily concave. It looks like outside the convex programming framework, all problems are treated the same. This paper is a companion paper to [10] where we proved that if the nondegeneracy condition

$$\forall j = 1, \ldots, m: \quad \nabla g_j(\mathbf{x}) \neq 0 \quad \forall \mathbf{x} \in \mathbf{K} \text{ with } g_j(\mathbf{x}) = 0 \tag{1.3}$$

holds, then $\mathbf{x} \in \mathbf{K}$ is a global minimizer of f on $\mathbf{K}$ if and only if $(\mathbf{x}, \lambda)$ is a KKT point for some $\lambda \in \mathbb{R}^m_+$. This indicates that for convex optimization problems (1.1), and from the point of view of "first-order optimality conditions", what really matters is the geometry of $\mathbf{K}$ rather than its representation. Indeed, for *any* representation (g_j) of $\mathbf{K}$ that satisfies the nondegeneracy condition (1.3), there is a one-to-one correspondence between global minimizers and KKT points.

But what about from a computational viewpoint? Of course, not all representations of $\mathbf{K}$ are equivalent since the ability (as well as the efficiency) of algorithms to obtain a KKT point of $\mathbf{P}$ will strongly depend on the representation (g_j) of $\mathbf{K}$ which is used. For example, algorithms that implement Lagrangian duality would require the (g_j) to be concave, those based on second-order methods would require all functions f and (g_j) to be twice continuous differentiable, self-concordance of a barrier function associated with a representation of $\mathbf{K}$ may or may not hold, etc.

When $\mathbf{K}$ is convex but not its representation (g_j), several situations may occur. In particular, the level set $\{\mathbf{x} : g_j(\mathbf{x}) \geq a_j\}$ may be convex for $a_j = 0$ but not for some other values of $a_j > 0$, in which case the g_j's are not even quasiconcave on $\mathbf{K}$, i.e., one may say that $\mathbf{K}$ is convex *by accident* for the value $\mathbf{a} = 0$ of the parameter $\mathbf{a} \geq 0$. One might think that in this situation, algorithms that generate a sequence of feasible points in the interior of $\mathbf{K}$ could run into problems to find a local minimum of f. If the $-g_j$'s are all quasiconvex on $\mathbf{K}$, we say that we are in the generic convex case because not only $\mathbf{K}$ but also all sets $\mathbf{K}_{\mathbf{a}} := \{\mathbf{x} : g_j(\mathbf{x}) \geq \mathbf{a}_j, j = 1, \ldots, m\}$ are convex. However, quasiconvex functions do not share some nice properties of the convex functions. In particular, (a) $\nabla g_j(\mathbf{x}) = 0$ does not imply that g_j reaches a local minimum at $\mathbf{x}$, (b) a local minimum is not necessarily global and (c), the sum of quasiconvex functions is not quasiconvex in general; see e.g. Crouzeix et al. [5, p. 65]. And so even in this case, for some minimization algorithms, convergence to a minimum of f on $\mathbf{K}$ might be problematic.

So an interesting issue is to determine whether there is an algorithm which converges to a global minimizer of a convex function f on $\mathbf{K}$, no matter if the representation of $\mathbf{K}$ is convex or not. Of course, in view of [10, Theorem 2.3], a sufficient condition is that this algorithm provides a sequence (or subsequence) of points $(\mathbf{x}_k, \lambda_k) \in \mathbb{R}^n \times \mathbb{R}^m_+$ converging to a KKT point of $\mathbf{P}$.

With $\mathbf{P}$ and a parameter $\mu > 0$, we associate the *log-barrier* function $\phi_\mu : \mathbf{K} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathbf{x} \mapsto \phi_\mu(\mathbf{x}) := \begin{cases} f(\mathbf{x}) - \mu \displaystyle\sum_{j=1}^{n} \ln g_j(\mathbf{x}), & \text{if } g_j(\mathbf{x}) > 0, \forall j = 1, \ldots, m \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.4}$$

By a *stationary point* $\mathbf{x} \in \mathbf{K}$ of ϕ_μ, we mean a point $\mathbf{x} \in \mathbf{K}$ with $g_j(\mathbf{x}) \neq 0$ for all $j = 1, \ldots, m$, and such that $\nabla\phi_\mu(\mathbf{x}) = 0$. Notice that in general and in contrast with the present paper, ϕ_μ (or more precisely $\psi_\mu := \mu\phi_\mu$) is usually defined for convex problems $\mathbf{P}$ where all the g_j's are concave; see e.g. Den Hertog [6] and for more details on the barrier functions and their properties, the interested reader is referred to Güller [7] and Güller and Tuncel [8].

Contribution. The purpose of this paper is to show that no matter which representation (g_j) of a convex set $\mathbf{K}$ (assumed to be compact) is used (provided it satisfies the nondegeneracy condition (1.3)), any sequence of stationary points $(\mathbf{x}_\mu)$ of ϕ_μ, $\mu \to 0$, has the nice property that each of its accumulation points is a KKT point of $\mathbf{P}$ and hence, a global minimizer of f on $\mathbf{K}$. Hence, to obtain the global minimum of a convex function on $\mathbf{K}$ it is enough to minimize the log-barrier function for increasing values of the parameter, for any representation of $\mathbf{K}$ that satisfies the nondegeneracy condition (1.3). Again and of course, the efficiency of the method will crucially depends on the representation of $\mathbf{K}$ which is used. For instance, in general ϕ_μ will not have the self-concordance property, crucial for efficiency.

Observe that at first glance this result is a little surprising because as we already mentioned, there are examples of sets $\mathbf{K}_\mathbf{a} := \{\mathbf{x} : g_j(\mathbf{x}) \geq a_j,\, j = 1, \ldots, m\}$ which are non convex for every $0 \neq \mathbf{a} \geq 0$ but $\mathbf{K} := \mathbf{K}_\mathrm{o}$ is convex (by accident!) and (1.3) holds. So inside $\mathbf{K}$ the level sets of the g_j's are not convex any more. Still, and even though the stationary points $\mathbf{x}_\mu$ of the associated log-barrier ϕ_μ are inside $\mathbf{K}$, all converging subsequences of a sequence $(\mathbf{x}_\mu)$, $\mu \to 0$, will converge to some global minimizer $\mathbf{x}^*$ of f on $\mathbf{K}$. In particular, if the global minimizer $\mathbf{x}^* \in \mathbf{K}$ is unique then the whole sequence $(\mathbf{x}_\mu)$ will converge. Notice that this happens even if the g_j's are not log-concave, in which case ϕ_μ may not be convex for all μ (e.g. if f is linear). So what seems to really matter is the fact that as μ decreases, the convex function f becomes more and more important in ϕ_μ, and also that the functions g_j which matter in a KKT point $(\mathbf{x}^*, \lambda)$ are those for which $g_j(\mathbf{x}^*) = 0$ (and so with convex associated level set $\{\mathbf{x} : g_j(\mathbf{x}) \geq 0\}$).

2. Main result

Consider the optimization problem (1.1) in the following context.

Assumption 1. The set $\mathbf{K}$ in (1.2) is convex and Slater's assumption holds. Moroever, the nondegeneracy condition

$$\nabla g_j(\mathbf{x}) \neq 0 \quad \forall\, \mathbf{x} \in \mathbf{K} \text{ such that } g_j(\mathbf{x}) = 0, \tag{2.1}$$

holds for every $j = 1, \ldots, m$.

Observe that when the g_j's are concave then the nondegeneracy condition (2.1) holds automatically. Recall that $(\mathbf{x}^*, \lambda) \in \mathbf{K} \times \mathbb{R}^m$ is a Karush-Kuhn-Tucker (KKT) point of $\mathbf{P}$ if

- $\mathbf{x} \in \mathbf{K}$ and $\lambda \geq 0$
- $\lambda_j g_j(\mathbf{x}^*) = 0$ for every $j = 1, \ldots, m$
- $\nabla f(\mathbf{x}^*) - \sum_{j=1}^m \lambda_j \nabla g_j(\mathbf{x}^*) = 0$.

We recall the following result from [10]:

Theorem 1 ([10])**.** *Let* $\mathbf{K}$ *be as in (1.2) and let Assumption 1 hold. Then* $\mathbf{x}$ *is a global minimizer of* f *on* $\mathbf{K}$ *if and only if there is some* $\lambda \in \mathbb{R}^m_+$ *such that* $(\mathbf{x}, \lambda)$ *is a KKT point of* $\mathbf{P}$.

The next result is concerned with the log-barrier ϕ_μ in (1.4).

Lemma 2. *Let* $\mathbf{K}$ *in (1.2) be convex and compact and assume that Slater's condition holds. Then for every* $\mu > 0$ *the log-barrier function* ϕ_μ *in (1.4) has at least one stationary point on* $\mathbf{K}$ *(which is a global minimizer of* ϕ_μ *on* $\mathbf{K}$*).*

Proof. Let f^* be the minimum of f on $\mathbf{K}$ and let $0 < \delta := \max_k[\max\{g_k(\mathbf{x}) : \mathbf{x} \in \mathbf{K}\}]$. As Slater's condition holds, let $\mathbf{x}_0 \in \mathbf{K}$ be such that $g_j(\mathbf{x}_0) > 0$ for every $j = 1, \ldots, m$. Let $\epsilon > 0$ be fixed such that $\epsilon < \min_j g_j(\mathbf{x}_0)$, and $f^* - \mu \ln \epsilon - m \ln \delta > \phi_\mu(\mathbf{x}_0)$, and let $\mathbf{K}_\epsilon := \{\mathbf{x} \in \mathbf{K} : g_j(\mathbf{x}) \geq \epsilon, \ j = 1, \ldots, m\}$. By this choice of ϵ, one has $\phi_\mu(\mathbf{x}) \geq \phi_\mu(\mathbf{x}_0)$ for every $\mathbf{x} \in \mathbf{K} \setminus \mathbf{K}_\epsilon$. Moreover as ϕ_μ is continuous on $\mathbf{K}_\epsilon$, its (global) minimum $\rho_\epsilon\,(< \phi_\mu(\mathbf{x}_0))$ is attained at some $\mathbf{x}_\epsilon \in \mathbf{K}_\epsilon$. Therefore $\mathbf{x}_\epsilon$ is a global minimizer of ϕ_μ on $\mathbf{K}$ and as $g_j(\mathbf{x}_\epsilon) > 0$ for every $j = 1, \ldots, m$, then necessarily $\nabla \phi_\mu(\mathbf{x}_\epsilon) = 0$. □

We now state our main result.

Theorem 3. *Let* $\mathbf{K}$ *in (1.2) be compact and let Assumption 1 hold true. For every* $\mu > 0$*, let* $\mathbf{x}_\mu \in \mathbf{K}$ *be a stationary point of* ϕ_μ.

Then every accumulation point $\mathbf{x}^* \in \mathbf{K}$ *of a sequence* $(\mathbf{x}_\mu) \subset \mathbf{K}$ *with* $\mu \to 0$*, is a global minimizer of* f *on* $\mathbf{K}$*, and if* $\nabla f(\mathbf{x}^*) \neq 0$*,* $\mathbf{x}^*$ *is a KKT point of* $\mathbf{P}$.

Proof. Let $\mathbf{x}_\mu \in \mathbf{K}$ be a stationary point of ϕ_μ, which by Lemma 2 is guaranteed to exist. So

$$\nabla \phi_\mu(\mathbf{x}_\mu) \,=\, \nabla f(\mathbf{x}_\mu) - \sum_{j=1}^m \frac{\mu}{g_j(\mathbf{x}_\mu)} \nabla g_j(\mathbf{x}_\mu) \,=\, 0. \tag{2.2}$$

As $\mu \to 0$ and $\mathbf{K}$ is compact, there exists $\mathbf{x}^* \in \mathbf{K}$ and a subsequence $(\mu_\ell) \subset \mathbb{R}_+$ such that $\mathbf{x}_{\mu_\ell} \to \mathbf{x}^*$ as $\ell \to \infty$. We need consider two cases:

Case when $g_j(\mathbf{x}^*) > 0, \forall j = 1, \ldots, m$. Then as f and g_j are continuously differentiable, $j = 1, \ldots, m$, taking limit in (2.2) for the subsequence (μ_ℓ), yields $\nabla f(\mathbf{x}^*) = 0$ which, as f is convex, implies that $\mathbf{x}^*$ is a global minimizer of f on $\mathbb{R}^n$, hence on $\mathbf{K}$.

Case when $g_j(\mathbf{x}^*) = 0$ *for some* $j \in \{1, \ldots, m\}$. Let $J := \{j : g_j(\mathbf{x}^*) = 0\} \neq \emptyset$. We next show that for every $j \in J$, the sequence of ratii $(\mu/g_j(\mathbf{x}_{\mu_\ell}))$, $\ell = 1, \ldots$, is bounded. Indeed let $j \in J$ be fixed arbitrary. Let $\mathbf{x}_0 \in \mathbf{K}$ be as in the proof of Lemma 2; then $\langle \nabla g_j(\mathbf{x}^*), \mathbf{x}_0 - \mathbf{x}^* \rangle > 0$. Indeed, as $\mathbf{K}$ is convex, $\langle \nabla g_j(\mathbf{x}^*), \mathbf{x}_0 + \mathbf{v} - \mathbf{x}^* \rangle \geq 0$ for all $\mathbf{v}$ in some small enough ball $\mathbf{B}(0, \rho)$ around the origin. So if $\langle \nabla g_j(\mathbf{x}^*), \mathbf{x}_0 - \mathbf{x}^* \rangle = 0$ then $\langle \nabla g_j(\mathbf{x}^*), \mathbf{v} \rangle \geq 0$ for all $\mathbf{v} \in \mathbf{B}(0, \rho)$, in contradiction with $\nabla g_j(\mathbf{x}^*) \neq 0$. Next,

$$\begin{aligned}
\langle \nabla f(\mathbf{x}_{\mu_\ell}), \mathbf{x}_0 - \mathbf{x}^* \rangle \quad &= \quad \sum_{k=1}^m \frac{\mu}{g_k(\mathbf{x}_{\mu_\ell})} \langle \nabla g_k(\mathbf{x}_{\mu_\ell}), \mathbf{x}_0 - \mathbf{x}^* \rangle \\
&\geq \quad \frac{\mu}{g_j(\mathbf{x}_{\mu_\ell})} \langle \nabla g_j(\mathbf{x}_{\mu_\ell}), \mathbf{x}_0 - \mathbf{x}^* \rangle, \quad \ell = 1, \ldots,
\end{aligned}$$

where the last inequality holds because all terms in the summand are nonnegative. Hence, taking limit as $\ell \to \infty$ yields

$$\langle \nabla f(\mathbf{x}^*), \mathbf{x}_0 - \mathbf{x}^* \rangle \,\geq\, \langle \nabla g_j(\mathbf{x}^*), \mathbf{x}_0 - \mathbf{x}^* \rangle \times \lim_{\ell\to\infty} \frac{\mu}{g_j(\mathbf{x}_{\mu_\ell})},$$

which, as $j \in J$ was arbitrary, proves the required boundedness.

So take a subsequence (still denoted $(\mu_\ell)_\ell$ for convenience) such that the ratii $\mu/g_j(\mathbf{x}_{\mu_\ell})$ converge for all $j \in J$, that is,

$$\lim_{\ell\to\infty} \frac{\mu}{g_j(\mathbf{x}_{\mu_\ell})} \,=\, \lambda_j \geq 0, \qquad \forall\, j \in J,$$

and let $\lambda_j := 0$ for every $j \notin J$, so that $\lambda_j g_j(\mathbf{x}^*) = 0$ for every $j = 1, \ldots, m$. Taking limit in (2.2) yields

$$\nabla f(\mathbf{x}^*) \,=\, \sum_{j=1}^m \lambda_j \, \nabla g_j(\mathbf{x}^*), \tag{2.3}$$

which shows that $(\mathbf{x}^*, \lambda) \in \mathbf{K} \times \mathbb{R}^m_+$ is a KKT point for $\mathbf{P}$. Finally, invoking Theorem 1, $\mathbf{x}^*$ is also a global minimizer of $\mathbf{P}$. □

2.1. **Discussion.** Theorem 3 states that one may obtain a global minimizer of f on $\mathbf{K}$, by looking at any limit point of a sequence of stationary points $(\mathbf{x}_\mu)$, $\mu \to 0$, of the log-barrier function ϕ_μ associated with a representation (g_j) of $\mathbf{K}$, provided that the representation satisfies the nondegeneracy condition (1.3). To us, this comes as a little surprise as the stationary points $(\mathbf{x}_\mu)$ are all inside $\mathbf{K}$, and there are examples of convex sets $\mathbf{K}$ with a representation (g_j) satisfying (1.3) and such that the level sets $\mathbf{K}_\mathbf{a} = \{\mathbf{x} : g_j(\mathbf{x}) \geq a_j\}$ with $a_j > 0$, are not convex! Even if f is convex, the log-barrier function ϕ_μ need not be convex; for instance if f is linear, $\nabla^2 \phi_\mu = -\mu \sum_j \nabla^2 \ln g_j$, and so if the g_j's are not log-concave then ϕ_μ may not be convex on $\mathbf{K}$ for every value of the parameter $\mu > 0$.

Example 1. Let $n = 2$ and $\mathbf{K}_a := \{\mathbf{x} \in \mathbb{R}^2 \,:\, g(\mathbf{x}) \geq a\}$ with $\mathbf{x} \mapsto g(\mathbf{x}) := 4 - ((x_1+1)^2 + x_2^2)((x_1-1)^2 + x_2^2)$, with $a \in \mathbb{R}$. The set $\mathbf{K}_a$ is convex only for those values of a with $a \leq 0$; see in Figure 1. It is even disconnected fort $a = 4$.

We might want to consider a generic situation, that is, when the set

$$\mathbf{K}_\mathbf{a} := \{\mathbf{x} \in \mathbb{R}^n \,:\, g_j(\mathbf{x}) \geq a_j, \quad j = 1, \ldots, m\},$$

is also convex for every positive vector $0 \leq \mathbf{a} = (a_j) \in \mathbb{R}^m$. This in turn would imply that the g_j are *quasiconcave*[2] on $\mathbf{K}$. In particular, if the nondegeneracy condition (1.3) holds on $\mathbf{K}$ and the g_j's are twice differentiable, then at most one eigenvalue of the Hessian $\nabla^2 g_j$ (and hence $\nabla^2 \ln g_j$) is possibly positive (i.e., $\ln g_j$ is *almost* concave). This is because for every $\mathbf{x} \in \mathbf{K}$ with $g_j(\mathbf{x}) = 0$, one has $\langle \mathbf{v}, \nabla^2 g_j(\mathbf{x}) \mathbf{v} \rangle \leq 0$ for all $\mathbf{v} \in \nabla g_j(\mathbf{x})^\perp$ (where $\nabla g_j(\mathbf{x})^\perp := \{\mathbf{v} : \langle \nabla g_j(\mathbf{x}), \mathbf{v} \rangle = 0\}$). However, even in this situation, the log-barrier function ϕ_μ may not be convex. On the other hand, $\ln g_j$ is "more" concave than g_j on $\operatorname{Int} \mathbf{K}$ because its Hessian $\nabla^2 g_j$ satisfies $g_j^2 \nabla^2 \ln g_j = g_j \nabla^2 g_j - \nabla g_j \, (\nabla g_j)^T$. But still, g_j might not be log-concave on $\operatorname{Int} \mathbf{K}$, and so ϕ_μ may not be convex at least for values of μ not too small (and for all values of μ if f is linear).

[2] Recall that on a convex set $O \subset \mathbb{R}^n$, a function $f : O \to \mathbb{R}$ is quasiconvex if the level sets $\{\mathbf{x} : f(\mathbf{x}) \leq r\}$ are convex for every $r \in \mathbb{R}$. A function $f : O \to \mathbb{R}$ is said to be quasiconcave if $-f$ is quasiconvex; see e.g. [5].

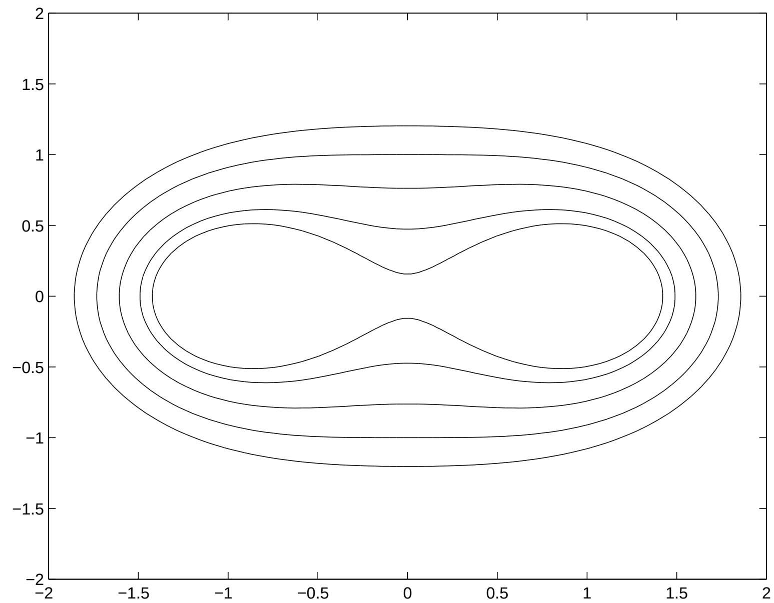

FIGURE 1. Example 1: Level sets $\{\mathbf{x} : g(\mathbf{x}) = a\}$ for $a = 2.95, 2.5, 1.5, 0$ and -2

Example 2. Let $n = 2$ and $\mathbf{K} := \{\mathbf{x} : g(\mathbf{x}) \geq 0, \mathbf{x} \geq 0\}$ with $\mathbf{x} \mapsto g(\mathbf{x}) = x_1x_2 - 1$. The representation of $\mathbf{K}$ is not convex but the g_j's are log-concave, and so the log-barrier $\mathbf{x} \mapsto \phi_\mu(\mathbf{x}) := f\mathbf{x}) - \mu(\ln g(\mathbf{x}) - \ln x_1 - \ln x_2)$ is convex.

Example 3. Let $n = 2$ and $\mathbf{K} := \{\mathbf{x} : g_1(\mathbf{x}) \geq 0;\, a - x_1 \geq 0;\, 0 \leq x_2 \leq b\}$ with $\mathbf{x} \mapsto g_1(\mathbf{x}) = x_1/(\epsilon + x_2^2)$ with $\epsilon > 0$. The representation of $\mathbf{K}$ is not convex and g_1 is not log-concave. If f is linear and ϵ is small enough, the log-barrier

$$\mathbf{x} \mapsto \phi_\mu(\mathbf{x}) := f(\mathbf{x}) - \mu(\ln x_1 + \ln(a - x_1) - \ln(\epsilon + x_2^2) + \ln x_2 + \ln(b - x_2))$$

is not convex for every value of $\mu > 0$.

We end up with a sufficient condition for the log-barrier ϕ_μ to be convex on $\mathbf{K}$.

Lemma 4. *Let $\mathbf{K}$ be as in (1.2), Assumption (1) hold and f be convex. Let $-g_j$ be twice differentiable and quasiconvex on $\mathbf{K}$ for every $j = 1, \ldots, m$. Then ϕ_μ is convex on $\mathbf{K}$ if for every $j = 1, \ldots, m$,*

$$g_j(\mathbf{x}) \left\langle \nabla g_j(\mathbf{x}), \nabla^2 g_j(\mathbf{x}) \nabla g_j(\mathbf{x}) \right\rangle \;\leq\; \|\nabla g_j(\mathbf{x})\|^4, \tag{2.4}$$

for all $\mathbf{x} \in \mathbf{K}$.

Proof. Observe that if $g_j(\mathbf{x}) \neq 0$ then $g_j^2 \nabla^2 \ln g_j = g_j \nabla^2 g_j - \nabla g_j\, (\nabla g_j)^T$. Moreover, for every $\mathbf{x} \in \mathbf{K}$, $\langle \mathbf{v}, \nabla^2 g_j(\mathbf{x})\mathbf{v}\rangle \leq 0$ for all $\mathbf{v} \in \nabla g_j(\mathbf{x})^\perp$, because $-g_j$ is quasiconvex on $\mathbf{K}$. And so if $g_j(\mathbf{x}) \neq 0$ then $\langle \mathbf{v}, \nabla^2 \ln g_j(\mathbf{x})\mathbf{v}\rangle \leq 0$ for all $\mathbf{v} \in \nabla g_j(\mathbf{x})^\perp$. Finally, for $\mathbf{v} := \nabla g_j(\mathbf{x})$ and $\mathbf{x} \in \mathbf{K}$ with $g_j(\mathbf{x}) \neq 0$, one has

$$-\langle \mathbf{v}, \nabla^2 \ln g_j(\mathbf{x})\mathbf{v}\rangle \;=\; -g_i(\mathbf{x}) \left\langle \nabla g_j(\mathbf{x}), \nabla^2 g_j(\mathbf{x}) \nabla g_j(\mathbf{x}) \right\rangle \;+\; \|\nabla g_j(\mathbf{x})\|^4,$$

which by (2.4) is nonnegative. Hence $-\nabla^2 \ln g_j(\mathbf{x})$ is positive semidefinite on $\mathbf{K} \setminus \{\mathbf{x} \in \mathbf{K} : g_j(\mathbf{x}) = 0\}$, which shows that $-\mu \sum_j \ln g_j$ is convex on $\Delta := \mathbf{K} \setminus \{\mathbf{x} \in \mathbf{K} : \exists j,\, g_j(\mathbf{x}) = 0\}$. As ϕ_μ is defined to be $+\infty$ on Δ, ϕ_μ is convex on $\mathbf{K}$. $\square$

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