Nonconvex Wireless Power Control

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CS 8292 : Advanced Topics in Convex Optimization and its Applications
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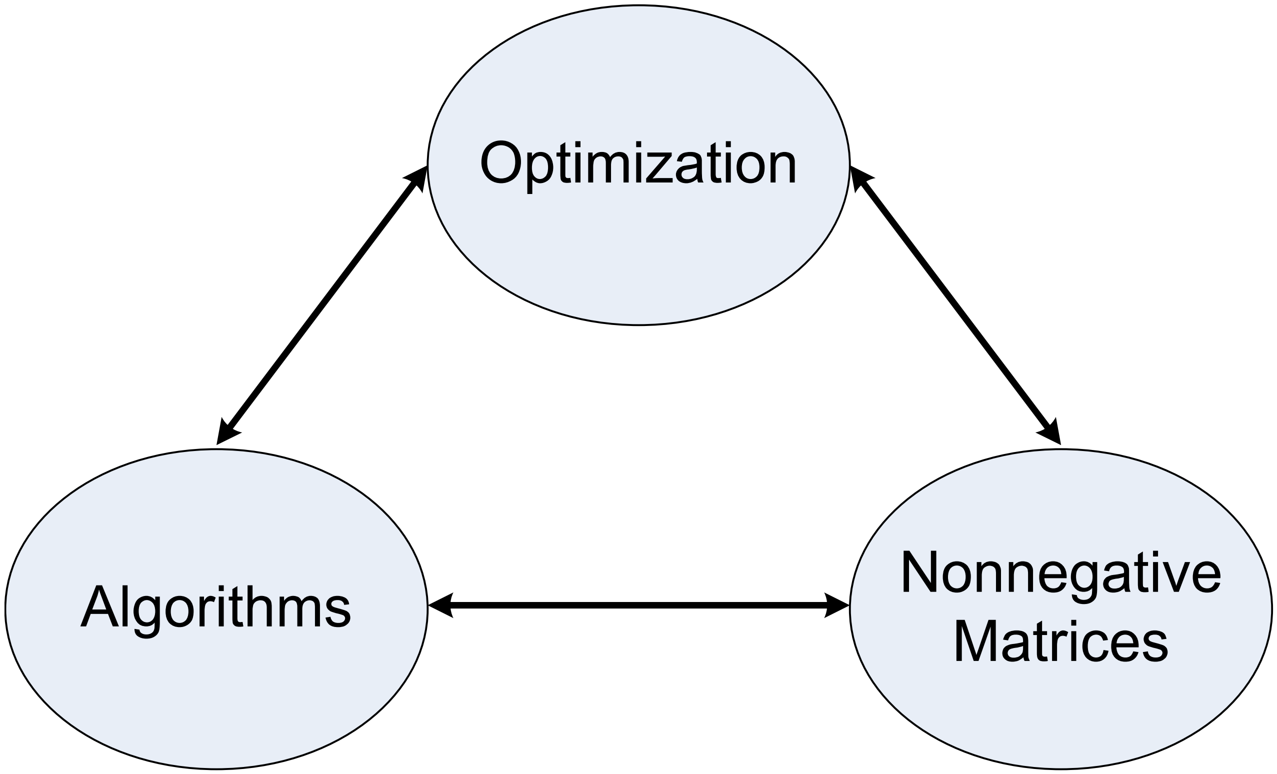
Outline

- Max-min weighted SIR optimization
- Nonlinear Perron-Frobenius Theorem and Algorithm
- Sum Rate Maximization

Interplay of Mathematical Tools



Max-Min Weighted SIR

- **Downlink** case: consider $\max_{\mathbf{p}\geq\mathbf{0}} \min_{l} \frac{\mathsf{SIR}_l(\mathbf{p})}{\beta_l}$ subject to $\sum_l p_l \leq \bar{P}$.

Theorem 1. *The optimal solution is such that the value* SIR_l/β_l *for all users are equal. The optimal weighted max-min* SIR *is given by*

$$\gamma^* = \frac{1}{\rho(\mathrm{diag}(\boldsymbol{\beta})\mathbf{B})},$$

where

$$\mathbf{B} = \mathbf{F} + (1/\bar{P})\mathbf{v}\mathbf{1}^\top$$

Further, the optimal $\mathbf{p}$*, denoted by* $\mathbf{p}^*$*, is* $t\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})\mathbf{B})$ *for some constant* $t > 0$*.*

C. W. Tan, M. Chiang & R. Srikant, , *Maximizing Sum Rate and Minimizing MSE on Multiuser Downlink: Optimality, Fast Algorithms, and Equivalence via Max-min SIR*, IEEE ISIT, 2009

Nonlinear Perron-Frobenius Theory

- Find $(\check{\lambda}, \check{\mathbf{s}})$ in

$$\lambda \mathbf{s} = \mathbf{A}\mathbf{s} + \mathbf{b}, \quad \lambda \in \mathbb{R}, \quad \mathbf{s} \geq \mathbf{0}, \quad \|\mathbf{s}\| = 1,$$

  where $\mathbf{A}$ and $\mathbf{b}$ is a square irreducible nonnegative matrix and nonnegative vector, respectively and $\|\cdot\|$ a monotone vector norm.

- $(\check{\lambda}, \check{\mathbf{s}})$ is Perron-Frobenius eigenvalue-vector pair of $\mathbf{A} + \mathbf{b}\mathbf{c}_*^\top$, where

$$\mathbf{c}_* = \arg \max_{\|\mathbf{c}\|_* = 1} \rho(\mathbf{A} + \mathbf{b}\mathbf{c}^\top),$$

  where $\|\cdot\|_*$ is the dual norm of $\|\cdot\|$, and $\check{\mathbf{s}} = (\mathbf{A}\check{\mathbf{s}} + \mathbf{b})/\|\mathbf{A}\check{\mathbf{s}} + \mathbf{b}\|$.

V. D. Blondel, L. Ninove and P. Van Dooren, *An affine eigenvalue problem on the nonnegative orthant*, Linear Algebra & its Applications, 2005

A Fast Max-min SIR Algorithm

- Based on Nonlinear Perron-Frobenius Theory

- **Algorithm 1.**

  1. *Update power* $\mathbf{p}(k+1)$*:*

  $$p_l(k+1) = \frac{\beta_l}{\mathsf{SIR}_l(\mathbf{p}(k))} p_l(k) \;\; \forall\, l.$$

  2. *Normalize* $\mathbf{p}(k+1)$*:*

  $$p_l(k+1) \leftarrow p_l(k+1) / \sum_j p_j(k+1) \cdot \bar{P} \;\; \forall\, l.$$

- **Theorem 2.** *Starting from any initial point* $\mathbf{p}(0)$*,* $\mathbf{p}(k)$ *in Algorithm 1 converges geometrically fast to* $\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})\mathbf{B})$ *(unique up to a scaling constant).*

Problem: Maximize Sum Shannon Rates

- Find $\mathbf{p}^\star = \arg \max_{\mathbf{0} \le \mathbf{p} \le \bar{\mathbf{p}}} \sum_l w_l \log(1 + \mathsf{SIR}_l(\mathbf{p}))$ where $\mathbf{1}^\top \mathbf{w} = 1$

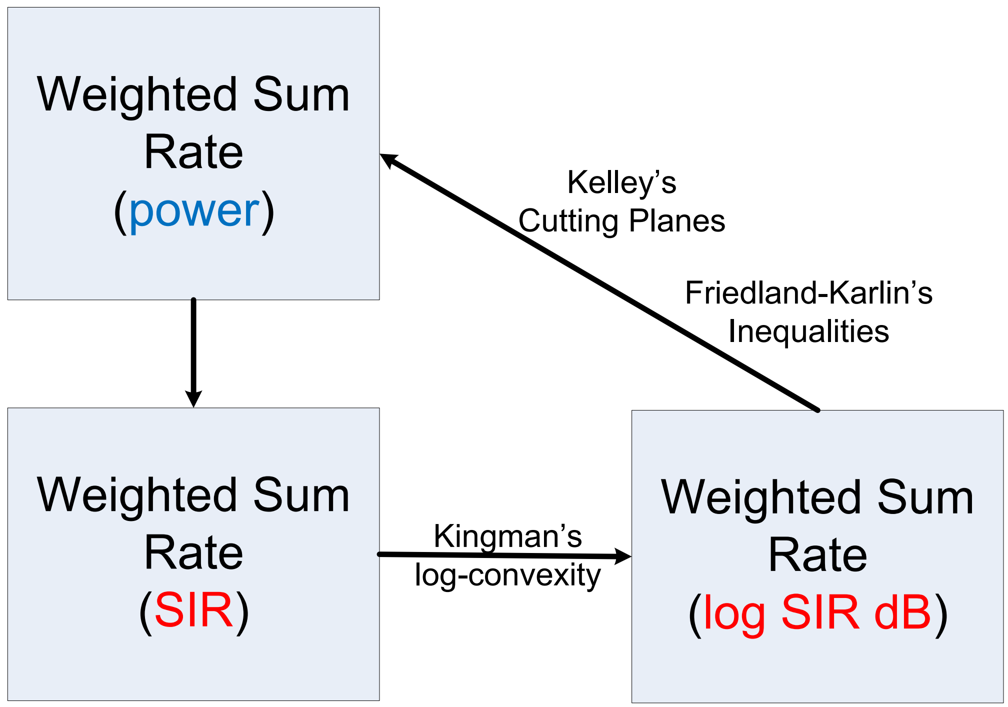
- Characterize the achievable rate region: $r_l = \log(1 + \mathsf{SIR}_l(\mathbf{p}))\ \forall\, l$

- Two-User case:

$$\max \quad w_1 \log\left(1 + \frac{G_{11} p_1}{G_{12} p_2 + n_1}\right) + w_2 \log\left(1 + \frac{G_{22} p_2}{G_{21} p_1 + n_2}\right)$$

subject to: $\quad 0 \le p_1 \le \bar{p}_1,\ \ 0 \le p_2 \le \bar{p}_2$

Solution Map



Tan, Friedland and Low, *Spectrum Management in Multiuser Cognitive Wireless Networks: Optimality and Algorithm*,

IEEE Journal on Selected Areas in Communications (JSAC), 2011

Sum Shannon Rate Global Optimization

- Nonlinear map between power $\mathbf{p}$ and SIR $\boldsymbol{\gamma} = \exp(\tilde{\boldsymbol{\gamma}})$:

$$\mathbf{p}^\star = (\mathbf{I} - \mathrm{diag}(\exp(\tilde{\boldsymbol{\gamma}}^\star))\mathbf{F})^{-1}\,\mathrm{diag}(\exp(\tilde{\boldsymbol{\gamma}}^\star))\mathbf{v} \tag{5}$$

- Constraints over $\mathbf{p} \leq \bar{\mathbf{p}}$ into $\tilde{\boldsymbol{\gamma}}$

- Convert into convex maximization (dB domain):

$$\begin{array}{rl} \text{maximize} & \sum_l w_l \log(1 + \exp(\tilde{\gamma}_l)) \\ \text{subject to} & \log \rho(\mathbf{diag}(\exp(\tilde{\boldsymbol{\gamma}}))(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top)) \leq 0 \quad \forall\, l, \\ \text{variables:} & \tilde{\gamma}_l, \quad \forall\, l. \end{array}$$

Nonnegative Matrix Theory: Minimax Theorem

- **Theorem 3.** *Friedland-Karlin inequality [**FriedlandKarlin'75**]: For any irreducible nonnegative matrix* $\mathbf{A}$*,*

$$\prod_l \left((\mathbf{A}\mathbf{z})_l / z_l\right)^{x_l y_l} \geq \rho(\mathbf{A})$$

for all strictly positive $\mathbf{z}$*, where* $\mathbf{x}$ *and* $\mathbf{y}$ *are the Perron and left eigenvectors of* $\mathbf{A}$ *respectively. Equality holds in (7) if and only if* $\mathbf{z} = a\mathbf{x}$ *for some positive* a*.*

- Donsker-Varadhan's variational principle (1975):

$$\max_{\boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{1}^\top \boldsymbol{\lambda} = 1} \min_{\mathbf{p} \geq \mathbf{0}} \sum_l \lambda_l \frac{(\mathbf{A}\mathbf{p})_l}{p_l} = \min_{\mathbf{p} \geq \mathbf{0}} \max_{\boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{1}^\top \boldsymbol{\lambda} = 1} \sum_l \lambda_l \frac{(\mathbf{A}\mathbf{p})_l}{p_l}$$

Sum Shannon Rate Global Optimization

- Convex Maximization (dB domain):

$$\begin{array}{rl} \text{maximize} & \sum_l w_l \log(1+\exp(\tilde{\gamma}_l)) \\ \text{subject to} & \log \rho(\mathsf{diag}(\exp(\tilde{\boldsymbol{\gamma}}))(\mathbf{F}+(1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top)) \le 0 \quad \forall\, l, \\ \text{variables:} & \tilde{\gamma}_l, \quad \forall\, l. \end{array}$$

- Relaxation of the constraint set by the Friedland-Karlin Inequalities:

$$\prod_l \gamma_l^{x_l(\mathbf{A})y_l(\mathbf{A})} \rho(\mathbf{A}) \le \rho(\mathsf{diag}(\boldsymbol{\gamma})\mathbf{A})$$

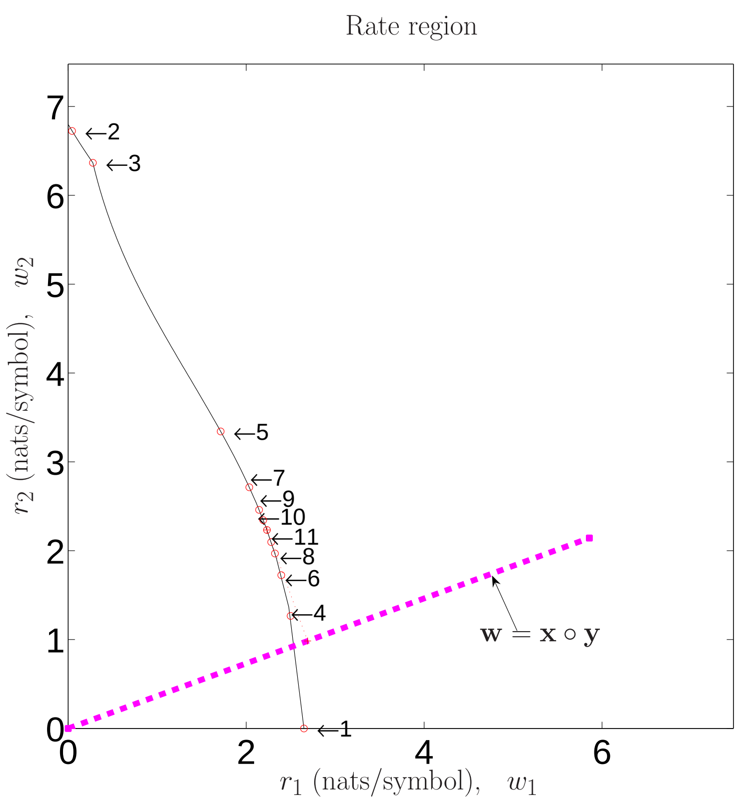
$$\sum_l x_l(\mathbf{A}) y_l(\mathbf{A}) \tilde{\gamma}_l + \log \rho(\mathbf{A}) \le \log \rho(\mathsf{diag}(\exp(\tilde{\boldsymbol{\gamma}}))\mathbf{A}) \quad (\text{dB domain}).$$

- Outer approximation algorithm (Kelley's cutting planes)

Global Optimizing Sum Rate: Examples

- $\lim_{k\to\infty} \min \left\{ \left(\mathbf{I} - \mathrm{diag}(\exp(\tilde{\boldsymbol{\gamma}}^k))\mathbf{F}\right)^{-1} \mathrm{diag}(\exp(\tilde{\boldsymbol{\gamma}}^k))\mathbf{v}, \bar{p}_l \right\} = \mathbf{p}^\star$

- Fast convergence in numerical examples



Global Optimizing Sum Rate: Examples

- Efficient and fast for small to medium-sized networks

Problem size	Maximal number of generated vertices	Number of iterations	CPU time (minutes)
2	15	12	0.062
4	139	760	4.1
6	14022	1238	83
8	283681	1968	468

Table 1: A comparison of the typical convergence and complexity statistics with the problem size. The CPU time is computed based on an implementation on a 64-bit Sun/Solaris 10 (SunOS 5.10) computer.

Summary

- Intriguing link between nonlinear Perron-Frobenius theory and geometric programming

- Distributed algorithm vs. centralized algorithm: configuration (no. of tuning parameters), feasibility, convergence, scalability etc.

Reading assignment:

- M. Chiang, P. Hande, T. Lan and C. W. Tan, Power Control by Geometric Programming, IEEE Trans. on Wireless Communications, 6(7), pp. 2640 - 2651, 2007.

- C. W. Tan, M. Chiang and R. Srikant, Maximizing Sum Rate and Minimizing MSE on Multiuser Downlink: Optimality, Fast Algorithms, and Equivalence via Max-min SIR, IEEE ISIT, 2009.
  See also *Convex Optimization* Textbook's Extra Exercises A12.3