Near Maximum Likelihood Detection Using An Interior Point Method and Semidefinite Programming

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Abstract

In this paper a maximum likelihood detection problem for a digital communication system is reformulated as a semidefinite programming (SDP) problem. A relaxation of this problem is done. An interior point method will be used to efficiently solve the semidefinite program arising from the relaxation. From the solution given by this interior point method, an approximate of the solution of the initial ML detection problem will be extracted using a randomization method. The detection method presented in this paper will have near ML performances with a polynomial complexity.

1. Introduction

Semidefinite programming involves the minimization of a linear function subject to the constraint that an affine combination of symmetric matrices is positive semidefinite. This constraint is in general nonlinear and nonsmooth yet convex. Semidefinite programming can be viewed as an extension of linear programming and reduces to the linear programming case when the symmetric matrices are diagonal. The two main areas of application for semidefinite programming are in combinatorial optimization and control theory.

Interior point methods are efficient methods to solve semidefinite programming problems. They were known as early as the 1960s in the form of the barrier function methods.

Here we will use semidefinite programming and interior point methods in the context of detection for a digital communication system. The problem of detection can be a problem of multiuser detection for a CDMA system or a detection problem for MIMO system.

2. System Model

Let a system sending an n-dimensional BPSK constellation that can be obtained from an $n/2$ two-dimensional QAM signal set by concatenating real and imaginary parts. The vector containing these BPSK symbols is indicated by $\mathbf{u} = (u_1,\ u_2,\ ...,\ u_n)^T$ where $u_i = \pm 1$. This constellation can be viewed as carved from a translated and scaled (enlarged by a factor of 2) version of the n-dimensional cubic lattice $\mathbb{Z}^n$. A rotated vector $\mathbf{x}$ is obtained by applying a rotation to the vector of data $\mathbf{u}$. So $\mathbf{x} = \mathbf{Mu}$ where $\mathbf{M}$ is the rotation matrix. This rotated vector is sent through a channel modelled as an independent Rayleigh fading channel. Hence we can write the received vector as

$$\mathbf{r} = \alpha \odot \mathbf{x} + \mathbf{w} \tag{1}$$

where $\mathbf{w} = (w_1,\ w_2,\ ...,\ w_n)^T$ is a gaussian noise vector, $\alpha = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_n)^T$ are random fading coefficients , and $\odot$ represent the component-wise product. In the equation (1), let the matrix $\mathbf{S} = Diag(\alpha)\mathbf{M}$. Hence $\mathbf{r} = \mathbf{Su} + \mathbf{n}$. So the output of the filter matched to $\mathbf{S}$ can be written as

$$\mathbf{y} = \mathbf{Hu} + \mathbf{z}, \tag{2}$$

where $\mathbf{H}$ is the correlation matrix equal to $\mathbf{S}^T\mathbf{S}$ and $\mathbf{z}$ is the gaussian noise vector with zero mean and autocorrelation matrix ($\sigma^2 = N_0/2$) $E[\mathbf{z}\ \mathbf{z}^T] = \sigma^2\mathbf{H}$.

In this work we will use $n \times n$ rotation matrices $\mathbf{M}$ generating $\mathbb{Z}_{n,n}$ lattices having maximum diversity equal to n.

3. Semidefinite programming

When the symbols u_i are equal to ± 1, to make a ML decision, we need to solve the optimization problem

$$\hat{\mathbf{u}} = arg \min_{\mathbf{u} \in \{\pm 1\}^n} \mathbf{u}^T\mathbf{Hu} - 2\mathbf{y}^T\mathbf{u} \tag{3}$$

The semidefinite programming is one of the most efficient methods to solve such optimization problem. The fundamentals of Semidefinite Programming are presented in [1], [2]. Let us start with the problem in (3). Introducing a redundant dummy variable u_{n+1}, we can express this equation as:

$$(\mathbf{u}^{*T}, u^*_{n+1}) = arg \min_{\mathbf{u}, u_{n+1}} [\mathbf{u}^T\ u_{n+1}] \begin{bmatrix} \mathbf{H} & -\mathbf{y} \\ -\mathbf{y}^T & 0 \end{bmatrix} \times \begin{bmatrix} \mathbf{u} \\ u_{n+1} \end{bmatrix} \quad s.t. \quad \begin{bmatrix} \mathbf{u} \\ u_{n+1} \end{bmatrix} \in \{\pm 1\}^{n+1},\ u_{n+1} = 1. \tag{4}$$

Based on (4), we reformulate (3) as

$$\mathbf{x}^* = arg \min_{\mathbf{x}} \quad \mathbf{x}^T\mathbf{Lx}\ s.t.\ \mathbf{x} \in \{-1, 1\}^{n+1}, \tag{5}$$

where

$$\mathbf{L} = \begin{bmatrix} \mathbf{H} & -\mathbf{y} \\ -\mathbf{y}^T & 0 \end{bmatrix}\ and\ \mathbf{x} = [\mathbf{u}^T\ u_{n+1}]^T.$$

For the semidefinite relaxation of (5), we observe that

$$\mathbf{x}^T\mathbf{L}\mathbf{x} = tr\{\mathbf{L}\mathbf{x}\mathbf{x}^T\}.$$

Since $\mathbf{x} \in \{-1, 1\}^{n+1}$, the matrix $\mathbf{x}\mathbf{x}^T$ is positive semidefinite where its diagonal elements are equal to 1. Its rank is equal to 1 [3]. Let $\mathbf{X} = \mathbf{x}\mathbf{x}^T$. We can rewrite the equation (5) as

$$\begin{aligned} \mathbf{X}_1^* &= arg\min_{\mathbf{X}} \quad tr\{\mathbf{L}\mathbf{X}\} \\ s.t.\ X_{ii} &= 1, \quad i = 1, ..., n+1, \quad rank(\mathbf{X}) = 1, \quad \mathbf{X} \succeq 0. \end{aligned} \tag{6}$$

$\mathbf{X} \succeq 0$ indicate that the matrix $\mathbf{X}$ is positive semidefinite. Due to the constraint $\mathbf{X} = \mathbf{x}\mathbf{x}^T$, the problem (6) is a nonconvex optimization problem. Now, if the rank= 1 constraint is removed from (6), we obtain the following relaxed problem:

$$\begin{aligned} \mathbf{X}^* &= arg\min_{\mathbf{X}} \quad tr\{\mathbf{L}\mathbf{X}\} \\ s.t.\ X_{ii} &= 1, \quad i = 1, ..., n+1, \quad \mathbf{X} \succeq 0. \end{aligned} \tag{7}$$

Problem (7) is known as a semidefinite relaxation programming (SDP) problem, and therefore, (7) is called an SD relaxation of (5). An advantage of using SD relaxation is that (7) is a convex optimization problem and, hence, does not suffer from local minima. Furthermore, an efficient optimization algorithm based on interior-point methods has been developed [4] for the SDP problem in (7).

4. Interior Point Method

For some standard minimization problem we use the so called barrier functions, penalizing points the more the closer they are to the specific boundary. For a sequence of points approaching the boundary the penalty will be steadily increasing and will tend to infinity on the boundary itself. This new unconstrained minimization problem is known as the barrier problem. Typically an interior point algorithm starts with a point somewhere in the middle, the interior, of the feasible region witch is here the cone of positive semidefinite matrices. It computes a descent direction with respect to the cost and penalty functions, and determines the next point by means of a line search. During this line search, the barrier functions guarantee that the iterates will always stay within the interior. However, optimal solutions of the constrained problems may and will in general be located on the boundary of the feasible region. Although it is impossible for an interior point algorithm to hit the boundary itself we can get a sequence of points converging to the optimal solution by solving a sequence of barrier problems. Whenever the current iterate is "close enough" to the optimal solution of the current barrier problem, the influence of the barrier functions is diminished and the next barrier problem is solved with the current iterate as starting point. The algorithm is stopped whenever the current iterate is "close enough" to the optimal solution of the original problem.

The interior point method of [4], [5] is designed to handle primal-dual pairs, where (7) is the primal problem and its dual problem is described as

$$\begin{aligned} (\mathbf{v}^*, \mathbf{Z}^*) = arg\max_{\mathbf{v},\mathbf{Z}} \quad \mathbf{e}_{n+1}^T\mathbf{v} \\ s.t\ \mathbf{Z} = \mathbf{L} - Diag(\mathbf{v}), \quad \mathbf{Z} \succeq 0. \end{aligned} \tag{8}$$

Where $\mathbf{e}_{n+1}$ the all ones vector of length $n+1$ and $Diag(\mathbf{v})$ denotes the diagonal matrix with diagonal elements obtained from $\mathbf{v}$.

The interior-point method starts off at an interior point of the cone of positive semidefinite matrices witch is a convex cone. The identity matrix $\mathbf{I}$ is contained in the cone and is a typical starting point. The iterates are the approximate minimizers of a sequence of auxiliary problems that contain additional term $(-\mu \log[det(\mathbf{X})])$ in the cost function. This term is called the barrier term and it consists of a barrier parameter $\mu > 0$ and a barrier function $-\log[det(\mathbf{X})]$. The auxiliary problem is then started as

$$\begin{aligned} \mathbf{X}_\mu = arg\ \min_{\mathbf{X}}\ tr\{\mathbf{L}\mathbf{X}\} - \mu \log[det(\mathbf{X})] \\ s.t.\ \ diag(\mathbf{X}) = \mathbf{e},\ (\mathbf{X} \succ 0). \end{aligned} \tag{9}$$

The barrier function grows to infinity as $\mathbf{X}$ approches the boundary of the semidenite cone. The barrier parameter μ controls the distance of the optimal solution of the auxiliary problem to the boundary. For a sequence of auxiliary barrier problems with $\mu \to 0$, the original cost function eventually dominates in the interior of the feasible set. It follows that the sequence of minimizers of the barrier problems converges to an optimizer of the problem (7).

We transform the barrier problem into an unconstrained problem for $(\mathbf{X} \succ 0)$ by introducing a Langrange multiplier vector $\mathbf{v}$ for the equality constraint,

$$L_\mu(\mathbf{X}, \mathbf{v}) = tr\{\mathbf{L}\mathbf{X}\} - -\mu \log[det(\mathbf{X})] + \mathbf{v}^T[\mathbf{e} - diag(\mathbf{X})].$$

For given $v \in \mathcal{R}^{n+1}$, the function $L_\mu(\mathbf{X}, \mathbf{v})$ is a smooth convex function. The first-order necessary conditions for optimality are called the Kurash-Kuhn-Tucker (KKT) conditions. By matrix calculus, the gradient of $\log[det(\mathbf{X})]$ is

$$\nabla_{\mathbf{X}} \log[det(\mathbf{X})] = \mathbf{X}^{-1}.$$

and we obtain the system

$$\begin{aligned} \nabla_{\mathbf{X}} L_\mu &= \mathbf{L} - \mu\mathbf{X}^{-1} - Diag(\mathbf{v}) = \mathbf{Z} - \mu\mathbf{X}^{-1} = \mathbf{0}, \\ \nabla_{\mathbf{v}} L_\mu &= \mathbf{a} - diag(\mathbf{X}) = \mathbf{0}. \end{aligned}$$

In a primal-dual formulation, we set $\mathbf{Z} = \mu\mathbf{X}^{-1}$ and rewrite KKT-conditions of the primal and dual barrier problems in the following form

$$\begin{aligned} diag(\mathbf{X}) &= \mathbf{a}, & \mathbf{X} \succ 0, \\ \mathbf{Z} + Diag(\mathbf{v}) &= \mathbf{L}, & \mathbf{Z} \succ 0, \\ \mathbf{X}\mathbf{Z} &= \mu\mathbf{I}. \end{aligned} \tag{10}$$

The first equation requires primal feasibility, the second requires dual feasibility, and the third equation is called perturbed complementarity condition.

We denote the solution of the equation system (10) for some fixed μ by $(\mathbf{X}_\mu^*, \mathbf{v}_\mu^*, \mathbf{Z}_\mu^*)$. $\mathbf{X}_\mu^*$ is the unique optimal

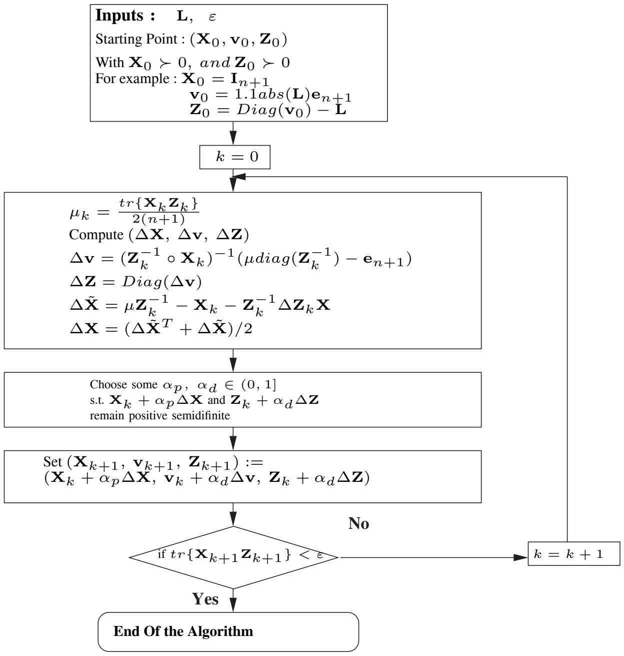

Fig. 1. A Flowchart illustrating the interior point algorithm for the semidefinite programming decoding

solution to the primal barrier problem, and $(\mathbf{v}_\mu^*, \mathbf{Z}_\mu^*)$ is the optimal solution to the analog dual barrier problem. $\mathbf{X}_\mu^*$ and $\mathbf{Z}_\mu^*$ are feasible primal and dual points of the original problem with a gap of $tr\{\mathbf{ZX}\} = (n+1)\mu$ between the objective values. The set of solutions $(\mathbf{X}_\mu^*, \mathbf{v}_\mu^*, \mathbf{Z}_\mu^*)$ for $\mu > 0$ forms the so-called central path, which is a smooth curve. It was shown in [4] that for $\mu \to 0$ the central path converges to a point $(\mathbf{X}^*, \mathbf{v}^*, \mathbf{Z}^*)$, where $\mathbf{X}^*$ is an optimal solution of the original primal problem (7) and $(\mathbf{v}^*, \mathbf{Z}^*)$ is an optimal solution of the original dual problem 8.
Here we give a flowchart explaining all the steps of the algorithm (See Fig. 1).

5. Approximate solution via Randomization

The interior point algorithm presented in the flowchart of Fig.4 give a solution of the relaxed problem (7). This solution is a matrix $\mathbf{X}^*$. So we must approximate a solution $\mathbf{x}^*$ to the original minimization problem (5) from $\mathbf{X}^*$. One method that do the extraction of $\mathbf{x}^*$ from $\mathbf{X}^*$ is presented in [6]. It is a randomization method. To understand intuitively the randomization, we consider an alternative expression of (5) and its semidefinite relaxation (7). We express (5) as:

$$\mathbf{x}^* = arg\min_{\mathbf{x}} \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} x_i x_j L_{ij} \quad s.t.\ \mathbf{x} \in \{-1, 1\}^{n+1} \quad (11)$$

The feasible matrix $\mathbf{X}$ for (7) can be factorized into $\mathbf{PP}^T$ where $\mathbf{P} = [\mathbf{p}_1, ..., \mathbf{p}_{n+1}]$. Each column has the norm $\|\mathbf{p}_i\| = 1$ because $diag(\mathbf{X}) = \mathbf{e}_{n+1}$. We may interpret $\mathbf{p}_i$ as a relaxation of $x_i \in \{-1, 1\}$ to the $(n+1)$-dimensional unit sphere [7]. The products $x_i x_j \in \{-1, 1\}$ are relaxed to $\mathbf{p}_i^T\mathbf{p}_j \in [-1, 1]$. Thus formulating the semidefinite relaxation (7) in vector notation we obtain [7]

$$\mathbf{P}^* = arg\min_{\mathbf{P}} \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} \mathbf{p}_i^T\mathbf{p}_j L_{ij} \quad s.t.\ \|\mathbf{p}_i\| = 1,\ i = 1, ..., n+1 \quad (12)$$

This vector formulation provides a clear intuition on how a semidefinite solution can be interpreted. Vectors $\mathbf{p}_i$ and $\mathbf{p}_j$ are unit vectors and $\mathbf{p}_i^T\mathbf{p}_j$ is the cosine of the angle defined by these vectors. If the angle between two vectors is large, then we should separate them in different sets. If it is small, we should put them in the same set. This is done by generating a random hyperplane through the origin and group all vectors on the same side of this hyperplane together. This hyperplane is constructed by a vector $\mathbf{q}$ acting as the normal vector of this hyperplane, where $\mathbf{q}$ is a random vector uniformly distributed on the $(n+1)$-dimensional unit sphere. The approximate of x_i^* is set to -1 if $\mathbf{p}_i^T\mathbf{q} < 0$ and 1 if otherwise. To further improve the approximation quality, the randomization algorithm is repeated a number of times, and the randomized solution with the smallest objective value is chosen as the approximate solution for $\mathbf{x}^*$.

6. Performance and Complexity

A. Performance of the detector

We can indicate that our problem of detection presented in the second section can be a multiuser detection problem applied for a CDMA system. It can be also a detection problem for a MIMO system. To evaluate the performances of this detector we run simulations for different value of n (the size of the problem). In Fig.2, the result of simulations for $n = 4$, $n = 6$ and $n = 8$ are presented. As we can see in this figure, the performances of the detector are much better than those of 4-QAM on a rayleigh fading channel. These performances are compared to those of two ML detectors which are the sphere decoder and the branch and bound detector [8]. This comparison is presented in Fig.3.

B. Complexity

We should remark that in the interior-point algorithm presented in the flowchart of Fig. 1 there is no expensive initialization and pre-decoding operations. But in the search process of this algorithm, there is some expensive ones. In fact, in each iteration we need to compute the matrix $\mathbf{Z}^{-1}$ and solve a liner system of size $n+1$. This will take $\mathcal{O}(n^3)$ operations of multiplication. For the line search, the algorithm must check of a Cholesky decomposition of $\mathbf{X}$ and $\mathbf{Z}$ exist. So the line search takes also $\mathcal{O}(n^3)$ operations. The algorithm therefore has a computational complexity witch is polynomial in n.

7. Conclusion

In this paper the problem of optimal ML detection is written as a semidefinite programming problem. A semidefinite relaxation is done for the latter. An interior point method with polynomial complexity is used to solve the semidefinite

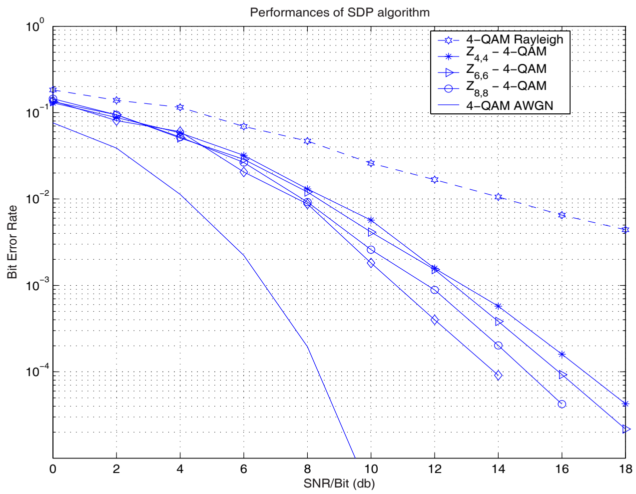

Fig. 2. Performances of The SDP algorithm on rayleigh fading channel for a system using 4-QAM Modulation : Without any precoding ($- \star -$), and with precoding using Lattice Constellations : $Z_{4,4}$ ($*$), $Z_{6,6}$ ($\triangleright$), $Z_{8,8}$ ($\circ$), $Z_{12,12}$ ($\Diamond$),Gaussian Channel ($-$).

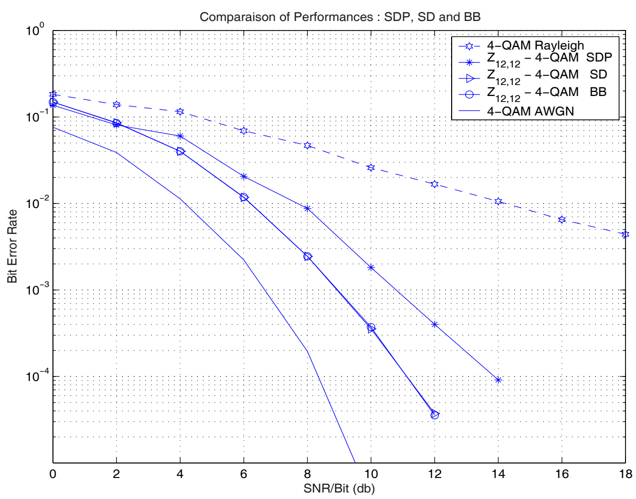

Fig. 3. Comparaison of the performances of the different algorithms (SDP ($*$), SD ($\triangleright$)and BB ($\diamond$)) on a rayleigh fading channel for a system using 4-QAM modulation an a precoding with a lattice constellation $Z_{12,12}$.

program arising from the relaxation. Finally a randomization algorithm which converts the solution of the relaxed problem to an approximate solution to the original problem is used in this work. Despite the fact that this method has little lower performances in comparison to those of the sphere decoding detection, it is a very interesting method to approach the ML performances with a complexity that is polynomial in n.
As a perspective to this work, we are searching how to apply this method to systems using modulations that have higher

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