LINEAR MATRIX INEQUALITY FORMULATION OF SPECTRAL MASK CONSTRAINTS

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ABSTRACT

The design of a finite impulse response filter often involves a spectral 'mask' which the magnitude spectrum must satisfy. This constraint can be awkward because it yields an infinite number of inequality constraints (two for each frequency point). In current practice, spectral masks are often approximated by discretization, but in this paper we will show that piecewise constant masks can be precisely enforced in a finite and convex manner via linear matrix inequalities. This facilitates the formulation of a diverse class of filter and beamformer design problems as semidefinite programmes. These optimization problems can be efficiently solved using recently developed interior point methods. Our results can be considered as extensions to the well-known Positive-Real and Bounded-Real Lemmas from the systems and control literature.

1. INTRODUCTION

In the design of finite impulse response (FIR) filters, one often encounters a spectral mask constraint on the magnitude of the frequency response of the filter (e.g., [1–4]). That is, for given $L(e^{j\omega})$ and $U(e^{j\omega})$, constrain the filter coefficients g_k so that

$$L(e^{j\omega}) \leq |G(e^{j\omega})| \leq U(e^{j\omega}) \qquad \text{for all } 0 \leq \omega < 2\pi, \qquad (1)$$

where $G(e^{j\omega}) = \sum_k g_k e^{-j\omega k}$, or determine that the constraint cannot be satisfied. A spectral mask constraint can be rather awkward to accommodate into general optimization-based filter design techniques because: (i) it is semi-infinite—there are two inequality constraints for every $\omega \in [0, 2\pi)$; and (ii) the set of feasible filter coefficients is in general non-convex due to the lower bound on $|G(e^{j\omega})|$. In order to efficiently solve filter design problems employing such constraints, we must find a way in which (1) can be represented in a finite and convex manner.

There are two established approaches to deal with the non-convexity of (1). The first is to enforce additional constraints on the parameters g_k so that $G(e^{j\omega})$ has 'linear phase'. In that case $|G(e^{j\omega})|$ becomes a linear function of approximately half the g_k's, and (1) can be reduced to two semi-infinite linear (and hence convex) constraints. The second is to reformulate (1) in terms of the autocorrelation of the filter $r_m = \sum_k g_k g_{k-m}$, [5, 6]. Since $R(e^{j\omega}) = |G(e^{j\omega})|^2$, Eq. (1) is equivalent to

$$L(e^{j\omega})^2 \leq R(e^{j\omega}) \leq U(e^{j\omega})^2 \qquad \text{for all } 0 \leq \omega < 2\pi, \qquad (2)$$

which amounts to two semi-infinite linear constraints on r_m. Note that the constraint that $R(e^{j\omega}) \geq L(e^{j\omega})^2 \geq 0$ is sufficient to ensure that a filter g_k can be extracted (though not uniquely) from a designed autocorrelation r_m via spectral factorization [6, 7].

The problem of representing (2) [or (1)] in a finite manner is more challenging. One approach is to approximate the constraints by discretizing them uniformly in frequency and enforce: $L(e^{j\omega_i})^2 + \varepsilon \leq R(e^{j\omega_i}) \leq U(e^{j\omega_i})^2 - \varepsilon$ for $\omega_i = 2\pi i/N$, $i = 0, 1, \ldots, N-1$, where ε and N are chosen heuristically. Unfortunately, as N is increased so that ε can be reduced, the resulting formulation can become prone to numerical difficulties. (Other discretization techniques are also available [8].) Algorithms of the exchange type [1, 3, 4] employ a non-uniform discretization of (2) at each stage of the algorithm, where the sample points are determined by the stationary points of the current estimate of the optimal $R(e^{j\omega})$. At each stage, an optimization problem is solved subject to appropriate equality constraints derived from (2) at those sample points. Although exchange methods often work well for the design of low-pass filters, substantial effort is required to guarantee the algorithm's convergence [4]. Furthermore, the algorithms may require significant 're-tailoring' in order to incorporate additional constraints on the filter coefficients (e.g., [9]).

The main result of this paper is that piecewise constant masks can be precisely represented in a convex and finite manner via a set of linear matrix inequalities (LMIs). (The reader is referred to [10] for the derivation and further references.) As a result, these masks can be efficiently incorporated, without discretization, into the diverse class of filter [6, 11, 12] and beamformer design algorithms based on convex optimization, and in particular semidefinite programming [13]. We will provide examples of effective design algorithms for standard linear-phase and nonlinear-phase FIR filters, and for narrow-band beamformers for linear antenna arrays.

When specialized to the case of a constant lower mask constraint, our results imply a new LMI formulation of the Positive Real Lemma [14] for FIR systems. This new formulation states that for r_m, $-M+1 \leq m \leq M-1$, with $r_{-m} = \bar{r}_m$, $R(e^{j\omega}) \geq 0$ for all $\omega \in [0, 2\pi]$ if and only if there exists an $M \times M$ positive semidefinite Hermitian matrix $\boldsymbol{X}$ such that $\mathrm{tr}(\boldsymbol{X}) = r_0$ and $\sum_{\ell=1}^{M-\ell} [\boldsymbol{X}]_{\ell+m,\ell} = r_m$, for $1 \leq m \leq M-1$. The more general results in this paper are LMI formulations of constraints of the form $R(e^{j\omega}) \geq L^2$ for all $\omega \in [\alpha, \beta]$, and therefore they can be considered as generalizations of the Positive Real Lemma. Simple manipulation of those results generates LMI formulations of constraints of the form $R(e^{j\omega}) \leq U^2$ for all $\omega \in [\alpha, \beta]$, which can be considered as generalizations of the Bounded Real Lemma.

Notation: Vectors and matrices will be represented by bold lowercase and uppercase letters, respectively, and their elements by medium weight lower case letters with appropriate subscripts; e.g., $g_k = [\boldsymbol{g}]_k$, $x_{ij} = [\boldsymbol{X}]_{ij}$. We define $\boldsymbol{v}(\theta; n) := \left[1, e^{j\theta}, e^{j2\theta}, \ldots, e^{jn\theta}\right]^T$, where "T" denotes the transpose, which is a basis for the (complex coefficient) trigonometric polynomials of

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degree n. If the sequence g_k denotes the impulse response of a causal FIR filter, and if $[\boldsymbol{g}]_k = g_k$, then the frequency response $G(e^{j\theta}) = \sum_{k=0}^{n} g_k e^{-jk\theta} = \boldsymbol{v}(\theta;n)^{H}\boldsymbol{g}$, where the superscript "H" denotes the conjugate transpose. We denote by $\mathscr{H}_+^{n\times n}$ the set of $n\times n$ positive semidefinite Hermitian matrices, and by $\mathscr{S}_+^{n\times n} \subset \mathscr{H}_+^{n\times n}$ the subset of real symmetric positive semidefinite matrices.

2. LINEAR MATRIX INEQUALITY FORMULATION

Given $0 \le \alpha < \beta < 2\pi$, we define the sets

$$\mathscr{K}(\alpha,\beta) := \{\boldsymbol{p} \in \mathbb{R}\times\mathbb{C}^n \mid \mathrm{Re}\, \boldsymbol{v}(\theta;n)^{H}\boldsymbol{p} \ge 0,\ \forall\theta \in [\alpha,\beta]\},$$
$$\bar{\mathscr{K}}(\alpha,\beta) := \{\boldsymbol{p} \in \mathbb{R}\times\mathbb{C}^n \mid \mathrm{Re}\, \boldsymbol{v}(\theta;n)^{H}\boldsymbol{p} \ge 0, \forall\theta \in [0,2\pi)\setminus(\alpha,\beta)\}.$$

These sets consist of the coefficients of the trigonometric polynomials whose real part is non-negative over a segment of the unit circle, or its complement, respectively. In this section we provide linear matrix inequality (LMI) descriptions of these sets—a result which may be of independent interest. Since the frequency response of a linear phase filter (or the autocorrelation of a general filter) can be written in terms of the real part of a trigonometric polynomial, these LMI descriptions of $\mathscr{K}(\alpha,\beta)$ and $\mathscr{K}(\alpha,\beta)$ generate a finite, convex formulation of piecewise constant spectral mask constraints, as will be shown in Section 3. For completeness, we let

$$\mathscr{K}(0,2\pi) := \{\boldsymbol{p} \in \mathbb{R}\times\mathbb{C}^n \mid \mathrm{Re}\, \boldsymbol{v}(\theta;n)^{H}\boldsymbol{p} \ge 0,\ \forall\theta \in [0,2\pi)\},$$

describe the trigonometric polynomials that are nonnegative on the entire unit circle.

In order to provide a concise LMI description of these cones, we define the unit lower triangular $(n+1)\times(n+1)$ Toeplitz matrices $\boldsymbol{T}_{0,n}, \boldsymbol{T}_{1,n}, \ldots, \boldsymbol{T}_{n,n}$ as

$$[\boldsymbol{T}_{k,n}]_{ij} = \begin{cases} 1, & \text{if } i = k+j, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{with } i,j \in \{0,1,\ldots,n\}.$$

Thus, $\boldsymbol{T}_{0,n} = \boldsymbol{I}$ and the matrix inner product $\boldsymbol{T}_{k,n}\bullet\boldsymbol{X} := \mathrm{tr}(\boldsymbol{T}_{k,n}^{T}\boldsymbol{X}) = \sum_{\ell=1}^{n+1-k}\boldsymbol{X}_{\ell+k,\ell}$, for all $\boldsymbol{X} \in \mathbb{C}^{(n+1)\times(n+1)}$. That is, $\boldsymbol{T}_{k,n}\bullet\boldsymbol{X}$ is the sum of the elements on the kth off-diagonal of $\boldsymbol{X}$. We define the linear operator $\check{\mathsf{L}}(\cdot)$ by $\boldsymbol{q} = \check{\mathsf{L}}(\boldsymbol{X}) \in \mathbb{C}^{n+1}$, where

$$q_0 = \boldsymbol{T}_{0,n}\bullet\boldsymbol{X}, \quad q_i = 2(\boldsymbol{T}_{i,n}\bullet\boldsymbol{X}), \quad \text{for } i = 1,2,\ldots,n.$$

Given $0 \le \alpha < \beta < 2\pi$, we define the vector $\boldsymbol{d}(\alpha,\beta) \in \mathbb{R}\times\mathbb{C}$ as:

$$\boldsymbol{d}(\alpha,\beta) := \begin{cases} \begin{bmatrix} \cos\alpha + \cos\beta - \cos(\beta-\alpha) - 1 \\ (1-e^{j\alpha})(e^{j\beta}-1) \end{bmatrix} & \text{if } \alpha > 0 \\ \begin{bmatrix} -\sin\beta \\ j(1-e^{j\beta}) \end{bmatrix} & \text{if } \alpha = 0. \end{cases}$$

Using $\boldsymbol{d}(\alpha,\beta)$, we define the linear operator $\check{\Lambda}(\cdot)$, by $\boldsymbol{q} = \check{\Lambda}(\boldsymbol{X};\alpha,\beta) \in \mathbb{C}^n$, where

$$\begin{aligned} q_0 &= d_0(\alpha,\beta)(\boldsymbol{T}_{0,n-1}\bullet\boldsymbol{X}) + \overline{d_1(\alpha,\beta)}(\boldsymbol{T}_{1,n-1}\bullet\boldsymbol{X}), \\ q_k &= 2d_0(\alpha,\beta)(\boldsymbol{T}_{k,n-1}\bullet\boldsymbol{X}) + d_1(\alpha,\beta)(\boldsymbol{T}_{k-1,n-1}\bullet\boldsymbol{X}) \\ &\quad + \overline{d_1(\alpha,\beta)}(\boldsymbol{T}_{k+1,n-1}\bullet\boldsymbol{X}),\ \text{for } k = 1,2,\ldots,n-2, \\ q_{n-1} &= 2d_0(\alpha,\beta)(\boldsymbol{T}_{n-1,n-1}\bullet\boldsymbol{X}) + d_1(\alpha,\beta)(\boldsymbol{T}_{n-2,n-1}\bullet\boldsymbol{X}), \\ q_n &= d_1(\alpha,\beta)(\boldsymbol{T}_{n-1,n-1}\bullet\boldsymbol{X}). \end{aligned}$$

With this notation, we can now state our main result:

Theorem 1 *For $0 \le \alpha < \beta < 2\pi$, the cones $\mathscr{K}(\alpha,\beta)$, $\bar{\mathscr{K}}(\alpha,\beta)$ and $\mathscr{K}(0,2\pi)$ admit the following LMI descriptions:*

$$\begin{aligned} \mathscr{K}(\alpha,\beta) &= \{\boldsymbol{p} \mid \boldsymbol{p} + \xi j\boldsymbol{e}_0 = \check{\mathsf{L}}(\boldsymbol{X}) + \check{\Lambda}(\boldsymbol{Z};\alpha,\beta),\ \textit{for some } \xi\in\mathbb{R}, \\ &\qquad \boldsymbol{X} \in \mathscr{H}_+^{(n+1)\times(n+1)}, \boldsymbol{Z} \in \mathscr{H}_+^{n\times n}\}, \\ \bar{\mathscr{K}}(\alpha,\beta) &= \{\boldsymbol{p} \mid \boldsymbol{p} + \xi j\boldsymbol{e}_0 = \check{\mathsf{L}}(\boldsymbol{X}) - \check{\Lambda}(\boldsymbol{Z};\alpha,\beta),\ \textit{for some } \xi\in\mathbb{R}, \\ &\qquad \boldsymbol{X} \in \mathscr{H}_+^{(n+1)\times(n+1)}, \boldsymbol{Z} \in \mathscr{H}_+^{n\times n}\}, \\ \mathscr{K}(0,2\pi) &= \{\check{\mathsf{L}}(\boldsymbol{X}) \mid \boldsymbol{X} \in \mathscr{H}_+^{(n+1)\times(n+1)}\}. \end{aligned} \tag{3}$$

Equation (3) is the new formulation of the Positive Real Lemma [14] for FIR systems stated in the Introduction. Thus, Theorem 1 can be seen as an extension of the Positive Real Lemma for FIR systems. For real trigonometric polynomials of the form $\sum_{k=0}^{n} p_k\cos(k\theta)$, the results in Theorem 1 may be simplified:

$$\begin{aligned} \mathscr{K}_{\mathrm{real}}(\alpha,\beta) &= \Big\{\boldsymbol{p} \in \mathbb{R}^{n+1} \,\Big|\, \sum_{k=0}^{n} p_k\cos(k\theta) \ge 0, \forall\theta\in[\alpha,\beta]\Big\} \\ &= \Big\{\boldsymbol{p} \,\Big|\, \boldsymbol{p} = \check{\mathsf{L}}(\boldsymbol{X}) + \check{\Lambda}(\boldsymbol{Z};\alpha,\beta), \quad \text{for some} \\ &\qquad \boldsymbol{X} \in \mathscr{S}_+^{(n+1)\times(n+1)}, \boldsymbol{Z} \in \mathscr{S}_+^{n\times n}\Big\}, \end{aligned} \tag{4}$$

and similarly for $\bar{\mathscr{K}}_{\mathrm{real}}(\alpha,\beta)$. Notice that $\boldsymbol{X}$ and $\boldsymbol{Z}$ in (4) are real symmetric, rather than complex Hermitian. Further (internal) simplifications of (4) are possible if the segment $[\alpha,\beta]$ is of the form $[\alpha,\pi]$, [10].

3. APPLICATIONS TO FIR FILTER DESIGN

In this section we use the results of Section 2 to precisely transform the piecewise constant portions of the masks in (1) or (2) into pairs of LMIs. The first step is to write $G(e^{j\theta})$ or $R(e^{j\theta})$ in the form $\mathrm{Re}\, \boldsymbol{v}(\theta;\cdot)^{H}\boldsymbol{p}$. To do so, we define $\boldsymbol{M} \in \mathbb{R}^{(2M-1)\times M}$ and $\tilde{\boldsymbol{I}} \in \mathbb{R}^{M\times M}$ such that

$$\boldsymbol{M} := \begin{bmatrix} \boldsymbol{0} & \boldsymbol{J} \\ 1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \qquad \text{and} \qquad \tilde{\boldsymbol{I}} := \begin{bmatrix} 1 & \boldsymbol{0} \\ \boldsymbol{0} & 2\boldsymbol{I} \end{bmatrix}, \tag{5}$$

where $\boldsymbol{I}$ is the $(M-1)\times(M-1)$ identity matrix and $\boldsymbol{J}$ is the $(M-1)\times(M-1)$ matrix with ones on the anti-diagonal and zeros elsewhere. For a filter of length M, if we define $\tilde{\boldsymbol{r}} \in \mathbb{R}^M$ such that $[\tilde{\boldsymbol{r}}]_m = r_m$, $0 \le m \le M-1$, then $R(e^{j\theta}) = e^{j(M-1)\theta}\boldsymbol{v}(\theta;2M-1)^{H}\boldsymbol{M}\tilde{\boldsymbol{r}} = \mathrm{Re}\, \boldsymbol{v}(\theta;M-1)^{H}\tilde{\boldsymbol{I}}\tilde{\boldsymbol{r}}$. Similarly, for a filter of odd length $2M-1$ which is symmetric and centered at the origin, if we define $\tilde{\boldsymbol{g}} \in \mathbb{R}^M$ such that $[\tilde{\boldsymbol{g}}]_k = g_k$, $0 \le k \le M-1$ then, $G(e^{j\theta}) = \mathrm{Re}\, \boldsymbol{v}(\theta;M-1)^{H}\tilde{\boldsymbol{I}}\tilde{\boldsymbol{g}}$. (Other linear phase filters can be handled in similar ways.)

We will focus on the design of a simple low-pass filter with a relative spectral mask $\zeta\bar{L}(e^{j\theta}) \le |G(e^{j\theta})| \le \zeta\bar{U}(e^{j\theta})$. The extension to band-pass and multi-band filter is immediate and is implicit in the design in Section 4. For a simple low-pass filter, the relative spectral mask has

$$\bar{L}(e^{j\theta}) = \begin{cases} \bar{L}_p & 0 \le \theta \le 2\pi f_p \\ \bar{L}_s & 2\pi f_p < \theta \le \pi \end{cases}, \quad \bar{U}(e^{j\theta}) = \begin{cases} \bar{U}_p & 0 \le \theta < 2\pi f_s \\ \bar{U}_s & 2\pi f_s \le \theta \le \pi \end{cases}, \tag{6}$$

with f_p and f_s denoting the normalized frequencies of the pass-band and stop-band edges, respectively, $0 \le f_p < f_s \le 1/2$, and

$\bar{U}_s \le \bar{U}_p$, $\bar{L}_p \le \bar{U}_p$ and $\bar{L}_s \le \bar{L}_p$. In the case of linear phase filters we set $\bar{L}_s = -\bar{U}_s$ and for autocorrelation designs we set $\bar{L}_s = 0$. By observing the common form of $G(e^{j\theta})$ and $R(e^{j\theta})$ above, and that $\bar{L}_s \le \bar{L}_p$ and $\bar{U}_s \le \bar{U}_p$, the spectral mask constraint can be re-written in a generic form as

$$\tilde{\boldsymbol{I}}\tilde{\boldsymbol{x}} - \zeta^q \bar{L}_p^q \boldsymbol{e}_0 \in \mathscr{K}_{\text{real}}(2\pi f_p, \pi), \quad \zeta^q \bar{U}_p^q \boldsymbol{e}_0 - \tilde{\boldsymbol{I}}\tilde{\boldsymbol{x}} \in \mathscr{K}_{\text{real}}(0, \pi),$$
$$\tilde{\boldsymbol{I}}\tilde{\boldsymbol{x}} - \zeta^q \bar{L}_s^q \boldsymbol{e}_0 \in \mathscr{K}_{\text{real}}(0, \pi), \qquad \zeta^q \bar{U}_s^q \boldsymbol{e}_0 - \tilde{\boldsymbol{I}}\tilde{\boldsymbol{x}} \in \mathscr{K}_{\text{real}}(2\pi f_s, \pi), \tag{7}$$

where $q = 1$ and $\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{g}}$ when we design an odd-length symmetric filter, and $q = 2$ and $\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{r}}$ for autocorrelation designs.

The constraints in (7) define the set of feasible filters. A large class of filter design objectives can be cast as the minimization of a convex quadratic function of the parameters over the feasible set [3, 4]. Filter design problems in this class take the following form: Given a positive semidefinite matrix $\boldsymbol{Q}$, a vector $\boldsymbol{l}$ and an integer $q \in \{1, 2\}$, find $\tilde{\boldsymbol{x}}$ achieving $\min_{\tilde{\boldsymbol{x}}} \tilde{\boldsymbol{x}}^T \boldsymbol{Q} \tilde{\boldsymbol{x}} - 2\boldsymbol{l}^T \tilde{\boldsymbol{x}}$ subject to (7) and a linear normalization constraint on either $\tilde{\boldsymbol{x}}$ or ζ, or show that none exist. This generic design problem can be solved by solving the following convex optimization problem:

Problem 1 *Given $\boldsymbol{Q} = \boldsymbol{L}\boldsymbol{L}^T$, $\boldsymbol{l}$, q, f_p, f_s, $\bar{L}_p$, $\bar{L}_s$, $\bar{U}_p$, $\bar{U}_s$, and M, find $\tilde{\boldsymbol{x}} \in \mathbb{R}^M$ achieving $\min \Gamma - 2\boldsymbol{l}^T \tilde{\boldsymbol{x}}$ over $\tilde{\boldsymbol{x}}$, $\zeta > 0$, $\boldsymbol{X}^{(pu)}, \boldsymbol{X}^{(pl)}, \boldsymbol{X}^{(sl)}, \boldsymbol{X}^{(su)} \in S_+^{M \times M}$ and $\boldsymbol{Z}^{(pl)}, \boldsymbol{Z}^{(su)} \in S_+^{(M-1)\times(M-1)}$, subject to $\|\boldsymbol{L}^T \tilde{\boldsymbol{x}}\|_2^2 \le \Gamma$,*

$$\tilde{\boldsymbol{I}}\tilde{\boldsymbol{x}} - \zeta^q \bar{L}_p^q \boldsymbol{e}_0 = \check{\mathsf{L}}(\boldsymbol{X}^{(pl)}) - \check{\Lambda}\big(\boldsymbol{Z}^{(pl)}; 2\pi f_p, 2\pi(1 - f_p)\big), \tag{8}$$
$$\tilde{\boldsymbol{I}}\tilde{\boldsymbol{x}} - \zeta^q \bar{L}_s^q \boldsymbol{e}_0 = \check{\mathsf{L}}(\boldsymbol{X}^{(sl)}), \tag{9}$$
$$\zeta^q \bar{U}_p^q \boldsymbol{e}_0 - \tilde{\boldsymbol{I}}\tilde{\boldsymbol{x}} = \check{\mathsf{L}}(\boldsymbol{X}^{(pu)}), \tag{10}$$
$$\zeta^q \bar{U}_s^q \boldsymbol{e}_0 - \tilde{\boldsymbol{I}}\tilde{\boldsymbol{x}} = \check{\mathsf{L}}(\boldsymbol{X}^{(su)}) + \check{\Lambda}\big(\boldsymbol{Z}^{(su)}; 2\pi f_s, 2\pi(1 - f_s)\big), \tag{11}$$

and one of the normalizations $\zeta = 1$ or $\boldsymbol{c}^T \tilde{\boldsymbol{x}} = 1$, for a given vector $\boldsymbol{c}$, or show that none exist.

In Problem 1, Eqs (8) and (9) enforce the lower bound constraint of the spectral mask, and Eqs (10) and (11) enforce the upper bound constraint. Problem 1 consists of a linear objective, linear equality constraints [(8)–(11)], a linear inequality constraint on ζ, a 'rotated' second-order cone constraint $\|\boldsymbol{L}^T \tilde{\boldsymbol{x}}\|_2^2 \le \Gamma$, and positive semi-definiteness constraints on the various $\boldsymbol{X}$ and $\boldsymbol{Z}$ matrices. Hence, Problem 1 is a convex symmetric cone programme (e.g., [15]), which can be efficiently solved using interior point methods. Furthermore, infeasibility can be reliably detected. If $\tilde{\boldsymbol{x}}$ represents the autocorrelation sequence of the filter, then an optimal filter can be obtained (though not uniquely) from the solution of Problem 1 by spectral factorization [6, 7].

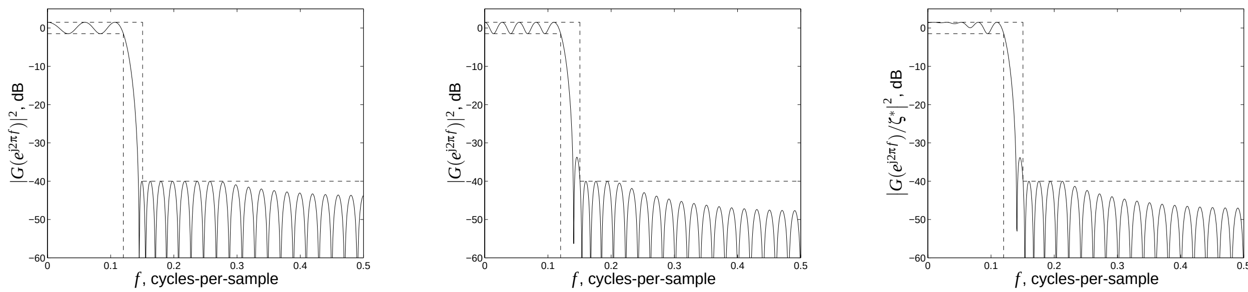
Example 1 Consider the design of a length 49 FIR filter which has the minimal 'stop-band energy', $E_s = (1/\pi) \int_{2\pi f_c}^{\pi} |G(e^{j\theta})|^2 \, d\theta$, subject to the spectral mask in (6), with $f_p = 0.1200$, $f_s = 0.1506$, $\bar{L}_p^2 = 10^{-0.15}$, $\bar{U}_p^2 = 10^{0.15}$, $\bar{U}_s^2 = 10^{-4}$. For an odd-length symmetric filter, $E_s = \tilde{\boldsymbol{g}}^T \boldsymbol{Q} \tilde{\boldsymbol{g}}$, where $\boldsymbol{Q} = \tilde{\boldsymbol{I}} \hat{\boldsymbol{Q}} \tilde{\boldsymbol{I}}$, $[\hat{\boldsymbol{Q}}]_{ij} = 2\big(\operatorname{sinc}(i + j) + \operatorname{sinc}(i - j)\big) - 4 f_c \big(\operatorname{sinc}(2 f_c (i + j)) + \operatorname{sinc}(2 f_c (i - j))\big)$, for $0 \le i, j \le M - 1$, and $\operatorname{sinc}(x) = \sin(\pi x)/(\pi x)$ for $x \ne 0$ and 1 for $x = 0$. For a general filter, $E_s = \tilde{\boldsymbol{l}}^T \tilde{\boldsymbol{r}}$, where $\tilde{l}_0 = 1/2 - f_c$ and $\tilde{l}_m = -2 f_c \operatorname{sinc}(2 f_c m)$, for $1 \le m \le M - 1$. Therefore, optimal filters can be designed using Problem 1 with the normalization constraint $\zeta = 1$. For $f_c = (f_p + f_s)/2$, the power spectrum of the optimal linear phase filter is shown in Fig. 1(a) and that of an optimal nonlinear phase filter is shown in Fig. 1(b). The linear-phase case was solved (using SeDuMi [15]) in 3.5 seconds on a 400 MHz PENTIUM II workstation, whilst the nonlinear phase case required 24 seconds. The sharper cut off and improved high-frequency decay of the nonlinear phase filter are clear from these figures. Although these filters minimize the stop-band energy, they do not minimize the proportion of the total energy of the filter in the stop band. A nonlinear phase filter which does so can be found by removing the constraint $\zeta = 1$ from Problem 1 (and hence allowing the mask to 'float'), and replacing it with $r_0 = 1$. The resulting optimal autocorrelation was obtained in 25 seconds and the power spectrum of an optimal filter is shown in Fig. 1(c). Observe that the flatter passband response in this case is achieved without greatly affecting the stop-band decay. □

We have also used a variant of Problem 1 to design robust 'chip' waveforms [10–12] for digital communication schemes based on code division multiple access, and to design (signal-independent and signal-adapted) filter banks and wavelets.

4. APPLICATIONS TO BEAMFORMER DESIGN

If $x_k(n)$ denotes (the complex envelope of) the output of the kth antenna in an array, and $\boldsymbol{w}$ is the vector of (conjugates of) the antenna weights, then the output of a narrow-band beamformer can be written as $y(n) = \boldsymbol{w}^H \boldsymbol{x}(n)$, [16]. It is well known [16], that the (complex) 'gain' in a direction at an angle ϕ to broadside of a linear equi-spaced array operating at wavelength λ with inter-element spacing $\lambda/2$ is $\tilde{W}(\phi) = e^{j\chi} W(e^{j\pi \sin\phi})$, where $W(e^{j\theta})$ is the Fourier Transform of w_k, and $e^{j\chi}$ determines the 'phase centre' of the array. In many applications, we would like to control the 'beam pattern' of the array, $|\tilde{W}(\phi)|^2$, but that results in non-convex constraints on w_k. Using the autocorrelation of the weights, $r_m = \sum_k w_k \bar{w}_{k-m}$, we have that $\tilde{R}(\phi) = \sum_m r_m e^{-jm\pi \sin\phi} = |\tilde{W}(\phi)|^2$, and therefore bound constraints on $|\tilde{W}(\phi)|^2$ result in linear constraints on r_m. For an M-element array, $\tilde{R}(\phi) = \operatorname{Re} \boldsymbol{v}(\pi \sin\phi; M - 1)^H \tilde{\boldsymbol{I}} \tilde{\boldsymbol{r}}$, where $[\tilde{\boldsymbol{r}}]_m = r_m$, $0 \le m \le M - 1$ and $\tilde{\boldsymbol{I}}$ was defined in Section 3. Therefore, piecewise constant constraints on $|\tilde{W}(\phi)|^2$ can be compactly enforced in an analogous way to that for the spectral masks in Section 3, as we now demonstrate in a simple example derived from [16, Fig. 2.5].

Example 2 Suppose that a desired signal impinges on a 16-element array of the above form from an angle of $\phi_d = -18^\circ \pm 6^\circ$, and that interfering signals arrive from angles in the range $\phi_i = 21.5^\circ \pm 6^\circ$. An interesting data independent [16] beamforming problem is to minimize the response to (spatially) white noise (i.e., $\boldsymbol{w}^H \boldsymbol{w} = r_0$), subject to the gain in the direction of the desired signal being within $\pm\Delta_d$ dB, and to the gain in the direction of the interferers being less than Δ_i dB. Furthermore, we would like to keep the sidelobes below Δ_s dB, and to constrain the main lobe to be within $-18^\circ \pm 13^\circ$. In short, our objective is to minimize the white noise gain, subject to a mask of the shape in Fig. 2(b). This problem can be cast in a similar way to Problem 1 with $\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{r}}$, $\zeta = 1$, $\boldsymbol{Q} = \boldsymbol{0}$, and $\boldsymbol{l} = -\boldsymbol{e}_0/2$, except that the vector $\tilde{\boldsymbol{r}}$ and the various $\boldsymbol{X}$ and $\boldsymbol{Z}$ matrices may now be complex. The trade-offs between the white noise gain and the level of interference suppression, for different values of the maximum sidelobe level and for a 'look direction ripple' $\Delta_d = 0.1$ dB can therefore be efficiently found. Some examples are shown in Fig. 2(a), and an example of the resulting beam patterns is shown in Fig. 2(b). (Each optimal $\tilde{\boldsymbol{r}}$ was computed in about seven seconds.) □

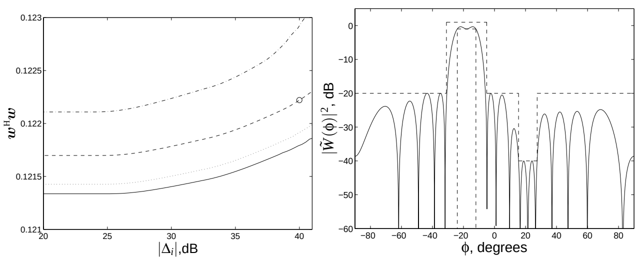


(a) Linear-phase filter minimizing stop-band energy.

(b) Nonlinear-phase filter minimizing stop-band energy.

(c) Nonlinear-phase filter minimizing proportion of energy in stop band.

Fig. 1. Power spectra of the filters from Ex. 1, along with the corresponding mask. In (c), ζ^* is the optimal value of ζ from Prob. 1.



(a) Tradeoffs

(b) Beam pattern

Fig. 2. (a) Trade-off between white noise gain, $\boldsymbol{w}^{\mathrm{H}}\boldsymbol{w}$, and interference suppression, $|\Delta_i|$, for sidelobe levels, $\Delta_s = 0.1$ dB (solid), -18 dB (dotted), -20 dB (dashed), and -22 dB (dot-dashed), for Ex. 2. The $\circ$ denotes the trade-off achieved by the beamformer in (b).

5. CONCLUDING REMARKS

In this paper, we have provided a compact representation of piecewise constant spectral mask constraints via linear matrix inequalities. This representation is precise and does not require discretization of the mask. The representation is also convex, which allows it to be incorporated into efficient design algorithms for a diverse class of FIR filters and beamformers. In addition to the applications considered here, generalizations of our results to rational filters (i.e., infinite impulse response filters) and to multidimensional filters are of interest in control theory, as well as signal and image processing, and are currently being pursued.

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