Advances in Convex Optimization: Interior-point Methods, Cone Programming, and Applications

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Easy and Hard Problems

Least squares (LS)

$$\text{minimize} \quad \|Ax - b\|_2^2$$

$A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$ are parameters; $x \in \mathbf{R}^n$ is variable

- have complete theory (existence & uniqueness, sensitivity analysis . . .)
- several algorithms compute (global) solution reliably
- can solve dense problems with $n = 1000$ vbles, $m = 10000$ terms
- by exploiting structure (e.g., sparsity) can solve **far larger** problems

. . . LS is a (widely used) **technology**

Linear program (LP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

$c,\ a_i \in \mathbf{R}^n$ are parameters; $x \in \mathbf{R}^n$ is variable

- have nearly complete theory
  (existence & uniqueness, sensitivity analysis . . .)
- several algorithms compute (global) solution reliably
- can solve dense problems with $n = 1000$ vbles, $m = 10000$ constraints
- by exploiting structure (e.g., sparsity) can solve **far larger** problems

. . . LP is a (widely used) **technology**

Quadratic program (QP)

$$
\begin{array}{ll}
\text{minimize} & \|Fx - g\|_2^2 \\
\text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m
\end{array}
$$

- a combination of LS & LP
- same story . . . QP is a technology
- solution methods reliable enough to be **embedded in real-time control applications** with little or no human oversight
- basis of **model predictive control**

The bad news

- LS, LP, and QP are **exceptions**
- most optimization problems, even some very simple looking ones, are **intractable**

Polynomial minimization

$$\text{minimize} \quad p(x)$$

p is polynomial of degree d; $x \in \mathbf{R}^n$ is variable

- except for special cases (e.g., $d = 2$) this is a **very difficult problem**
- even sparse problems with size $n = 20$, $d = 10$ are essentially intractable
- all algorithms known to solve this problem require effort exponential in n

What makes a problem easy or hard?

classical view:

- **linear** is easy
- **nonlinear** is hard(er)

What makes a problem easy or hard?

emerging (and correct) view:

. . . the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity.

— R. Rockafellar, SIAM Review 1993

Convex optimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_1(x) \leq 0, \ldots, f_m(x) \leq 0 \end{array}$$

$x \in \mathbf{R}^n$ is optimization variable; $f_i : \mathbf{R}^n \to \mathbf{R}$ are **convex**:

$$f_i(\lambda x + (1-\lambda)y) \leq \lambda f_i(x) + (1-\lambda) f_i(y)$$

for all x, y, $0 \leq \lambda \leq 1$

- includes LS, LP, QP, and **many others**
- like LS, LP, and QP, convex problems are **fundamentally tractable**

Example: Robust LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{Prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \ldots, m \end{array}$$

coefficient vectors a_i IID, $\mathcal{N}(\overline{a}_i, \Sigma_i)$; η is required reliability

- for fixed x, $a_i^T x$ is $\mathcal{N}(\overline{a}_i^T x, x^T \Sigma_i x)$
- so for $\eta = 50\%$, robust LP reduces to LP

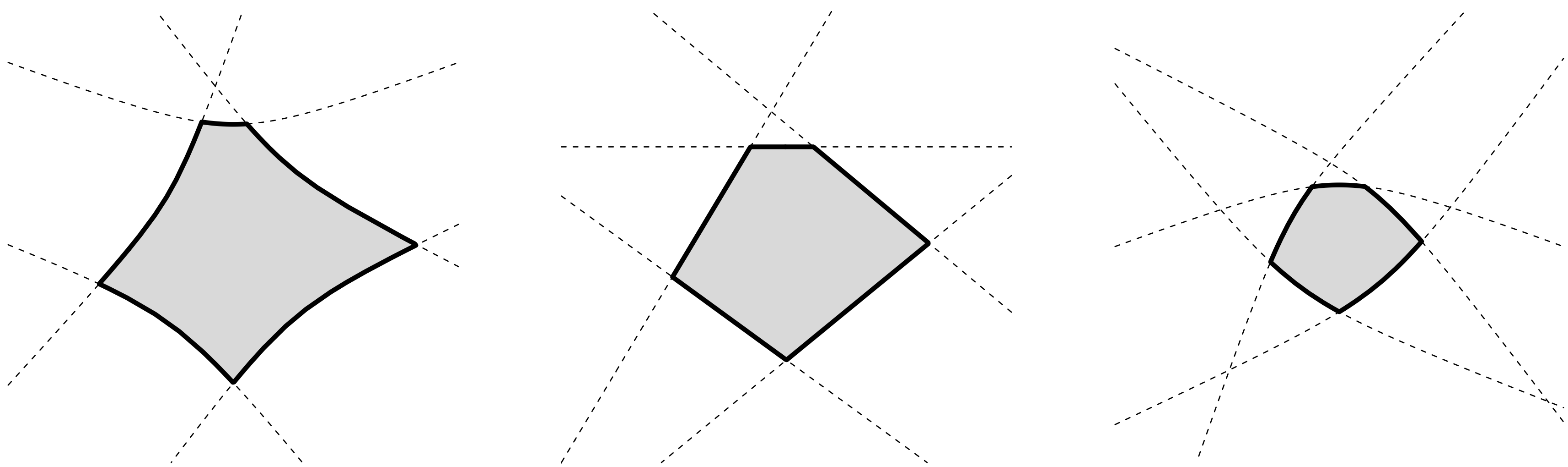
$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \overline{a}_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

  and so is easily solved

- what about other values of η, *e.g.*, $\eta = 10\%$? $\eta = 90\%$?

Hint

$$\{x \mid \mathbf{Prob}(a_i^T x \leq b_i) \geq \eta,\ i = 1, \ldots, m\}$$



$\eta = 10\%$ $\eta = 50\%$ $\eta = 90\%$

That's right

robust LP with reliability $\eta = 90\%$ is convex, and **very easily solved**

robust LP with reliability $\eta = 10\%$ is not convex, and **extremely difficult**

moral: **very difficult** and **very easy** problems can look **quite similar** (to the untrained eye)

Convex Analysis and Optimization

Convex analysis & optimization

nice properties of convex optimization problems known since 1960s

- local solutions are global
- duality theory, optimality conditions
- simple solution methods like alternating projections

convex analysis well developed by 1970s *Rockafellar*

- separating & supporting hyperplanes
- subgradient calculus

What's new (since 1990 or so)

- primal-dual interior-point (IP) methods
  extremely efficient, handle nonlinear large scale problems, polynomial-time complexity results, software implementations

- new standard problem classes
  generalizations of LP, with theory, algorithms, software

- extension to generalized inequalities
  semidefinite, cone programming

. . . convex optimization is becoming a **technology**

Applications and uses

- lots of applications
  control, combinatorial optimization, signal processing, circuit design, communications, . . .

- robust optimization
  robust versions of LP, LS, other problems

- relaxations and randomization
  provide bounds, heuristics for solving hard problems

Recent history

- 1984–97: interior-point methods for LP
  - 1984: Karmarkar's interior-point LP method
  - theory *Ye, Renegar, Kojima, Todd, Monteiro, Roos, . . .*
  - practice *Wright, Mehrotra, Vanderbei, Shanno, Lustig, . . .*
- 1988: Nesterov & Nemirovsky's self-concordance analysis
- 1989–: LMIs and semidefinite programming in control
- 1990–: semidefinite programming in combinatorial optimization *Alizadeh, Goemans, Williamson, Lovasz & Schrijver, Parrilo, . . .*
- 1994: interior-point methods for nonlinear convex problems *Nesterov & Nemirovsky, Overton, Todd, Ye, Sturm, . . .*
- 1997–: robust optimization *Ben Tal, Nemirovsky, El Ghaoui, . . .*

New Standard Convex Problem Classes

Some new standard convex problem classes

- second-order cone program (SOCP)
- geometric program (GP) (and entropy problems)
- semidefinite program (SDP)

for these new problem classes we have

- complete duality theory, similar to LP
- good algorithms, and robust, reliable software
- wide variety of new applications

Second-order cone program

second-order cone program (SOCP) has form

$$\begin{array}{ll} \text{minimize} & c_0^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m \end{array}$$

with variable $x \in \mathbf{R}^n$

- includes LP and QP as special cases
- nondifferentiable when $A_i x + b_i = 0$
- new IP methods can solve (almost) as fast as LPs

Example: robust linear program

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{Prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \ldots, m \end{array}$$

where $a_i \sim \mathcal{N}(\bar{a}_i, \Sigma_i)$

equivalent to

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \Phi^{-1}(\eta) \|\Sigma_i^{1/2} x\|_2 \leq 1, \quad i = 1, \ldots, m \end{array}$$

where Φ is (unit) normal CDF

robust LP is an SOCP for $\eta \geq 0.5$ ($\Phi(\eta) \geq 0$)

Geometric program (GP)

log-sum-exp function:

$$\mathbf{lse}(x) = \log\left(e^{x_1} + \cdots + e^{x_n}\right)$$

. . . a smooth **convex** approximation of the max function

geometric program:

$$\begin{array}{ll} \text{minimize} & \mathbf{lse}(A_0 x + b_0) \\ \text{subject to} & \mathbf{lse}(A_i x + b_i) \leq 0, \quad i = 1, \ldots, m \end{array}$$

$A_i \in \mathbf{R}^{m_i \times n}$, $b_i \in \mathbf{R}^{m_i}$; variable $x \in \mathbf{R}^n$

Entropy problems

unnormalized negative entropy is convex function

$$-\mathbf{entr}(x) = \sum_{i=1}^{n} x_i \log(x_i/\mathbf{1}^T x)$$

defined for $x_i \geq 0$, $\mathbf{1}^T x > 0$

entropy problem:

$$\begin{array}{ll} \text{minimize} & -\mathbf{entr}(A_0 x + b_0) \\ \text{subject to} & -\mathbf{entr}(A_i x + b_i) \leq 0, \quad i = 1, \ldots, m \end{array}$$

$A_i \in \mathbf{R}^{m_i \times n}$, $b_i \in \mathbf{R}^{m_i}$

Solving GPs (and entropy problems)

- GP and entropy problems are **duals** (if we solve one, we solve the other)
- new IP methods can solve large scale GPs (and entropy problems) almost as fast as LPs
- applications in many areas:
  - information theory, statistics
  - communications, wireless power control
  - digital and analog circuit design

CMOS analog/mixed-signal circuit design via GP

given

- **circuit cell**: *opamp, PLL, D/A, A/D, SC filter, . . .*
- **specs**: *power, area, bandwidth, nonlinearity, settling time, . . .*
- **IC fabrication process:** *TSMC* $0.18\mu\mathrm{m}$ *mixed-signal, . . .*

find

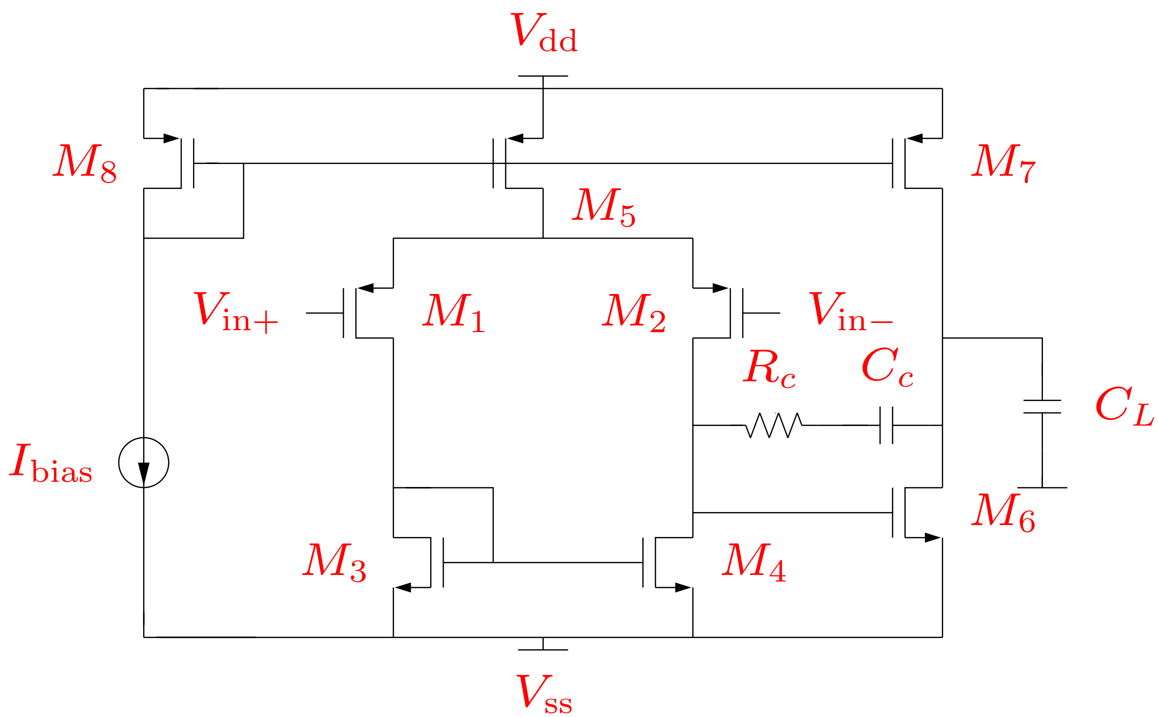
- **electronic design**: *device* L *&* W*, bias* I *&* V*, component values, . . .*
- **physical design**: *placement, layout, routing, GDSII, . . .*

The challenges

- complex, multivariable, highly nonlinear problem
- dominating issue: **robustness** to
  - model errors
  - parameter variation
  - unmodeled dynamics

(sound familiar?)

Two-stage op-amp



- **design variables:** *device lengths & widths, component values*
- **constraints/objectives**: *power, area, bandwidth, gain, noise, slew rate, output swing, . . .*

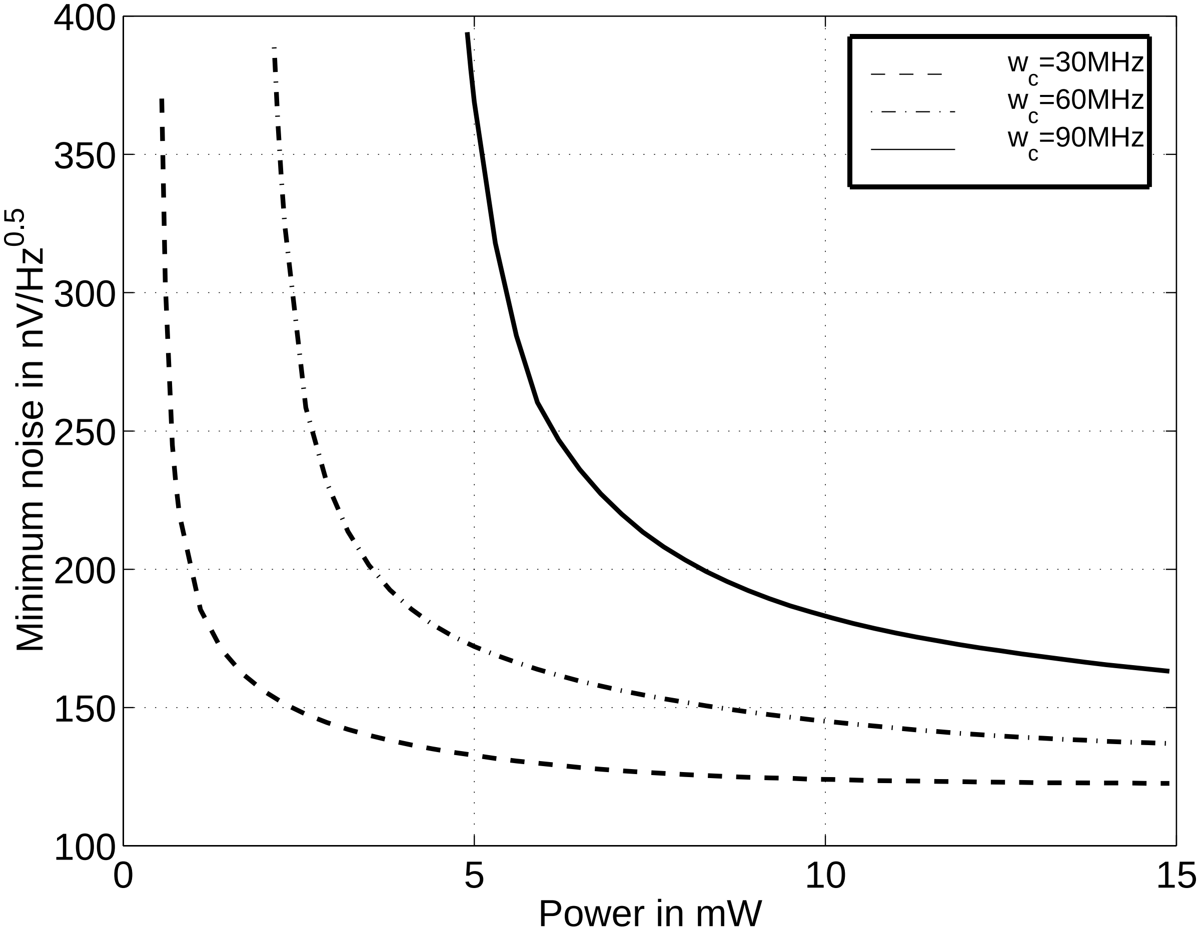
Op-amp design via GP

- express design problem as GP
  (using change of variables, and a few good approximations . . .)
- 10s of vbles, 100s of constraints; solution time $\ll 1$sec

robust version:

- take 10 (or so) different parameter values ('PVT corners')
- replicate all constraints for each parameter value
- get 100 vbles, 1000 constraints; solution time ≈ 2sec

Minimum noise versus power & BW



Cone Programming

Cone programming

general cone program:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq_K b \end{array}$$

- **generalized inequality** $Ax \preceq_K b$ means $b - Ax \in K$, a proper convex cone
- LP, QP, SOCP, GP can be expressed as cone programs

Semidefinite program

semidefinite program (SDP):

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 A_1 + \cdots + x_n A_n \preceq B \end{array}$$

$B, \; A_i$ are symmetric matrices; variable is $x \in \mathbf{R}^n$

- constraint is **linear matrix inequality** (LMI)
- inequality is matrix inequality, i.e., K is positive semidefinite cone
- SDP is special case of cone program

Early SDP applications

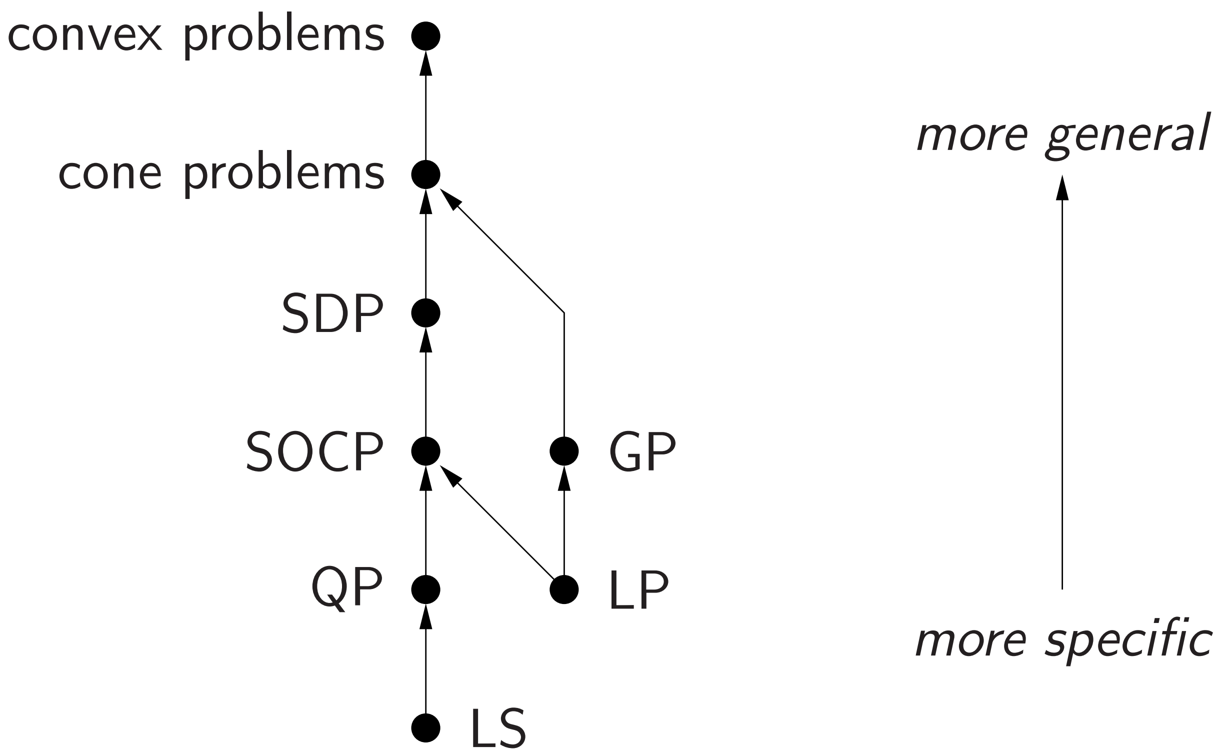
(around 1990 on)

- control *(many)*
- combinatorial optimization & graph theory *(many)*

More recent SDP applications

- structural optimization: *Ben-Tal, Nemirovsky, Kocvara, Bendsoe, . . .*
- signal processing: *Vandenberghe, Stoica, Lorenz, Davidson, Shaked, Nguyen, Luo, Sturm, Balakrishnan, Saadat, Fu, de Souza, . . .*
- circuit design: *El Gamal, Vandenberghe, Boyd, Yun, . . .*
- algebraic geometry:
  Parrilo, Sturmfels, Lasserre, de Klerk, Pressman, Pasechnik, . . .
- communications and information theory:
  Rasmussen, Rains, Abdi, Moulines, . . .
- quantum computing:
  Kitaev, Waltrous, Doherty, Parrilo, Spedalieri, Rains, . . .
- finance: *Iyengar, Goldfarb, . . .*

Convex optimization heirarchy



Relaxations & Randomization

Relaxations & randomization

convex optimization is increasingly used

- to find good bounds for hard (i.e., nonconvex) problems, via **relaxation**
- as a heuristic for finding good suboptimal points, often via **randomization**

Example: Boolean least-squares

Boolean least-squares problem:

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|^2 \\ \text{subject to} & x_i^2 = 1, \quad i = 1, \ldots, n \end{array}$$

- basic problem in digital communications
- could check all 2^n possible values of x . . .
- an NP-hard problem, and very hard in practice
- many heuristics for approximate solution

Boolean least-squares as matrix problem

$$\begin{aligned} \|Ax-b\|^2 &= x^TA^TAx - 2b^TAx + b^Tb \\ &= \mathbf{Tr}\, A^TAX - 2b^TA^Tx + b^Tb \end{aligned}$$

where $X = xx^T$

hence can express BLS as

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}\, A^TAX - 2b^TAx + b^Tb \\ \text{subject to} & X_{ii} = 1, \quad X \succeq xx^T, \quad \text{rank}(X) = 1 \end{array}$$

. . . still a very hard problem

SDP relaxation for BLS

ignore rank one constraint, and use

$$X \succeq xx^T \iff \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \succeq 0$$

to obtain **SDP relaxation** (with variables X, x)

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}\, A^T A X - 2b^T A^T x + b^T b \\ \text{subject to} & X_{ii} = 1, \quad \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \succeq 0 \end{array}$$

- optimal value of SDP gives **lower bound** for BLS
- if optimal matrix is rank one, we're done

Interpretation via randomization

- can think of variables X, x in SDP relaxation as defining a normal distribution $z \sim \mathcal{N}(x, X - xx^T)$, with $\mathbf{E}\, z_i^2 = 1$
- SDP objective is $\mathbf{E}\, \|Az - b\|^2$

suggests randomized method for BLS:

- find $X^\star$, $x^\star$, optimal for SDP relaxation
- generate z from $\mathcal{N}(x^\star, X^\star - x^\star x^{\star T})$
- take $x = \mathbf{sgn}(z)$ as approximate solution of BLS
(can repeat many times and take best one)

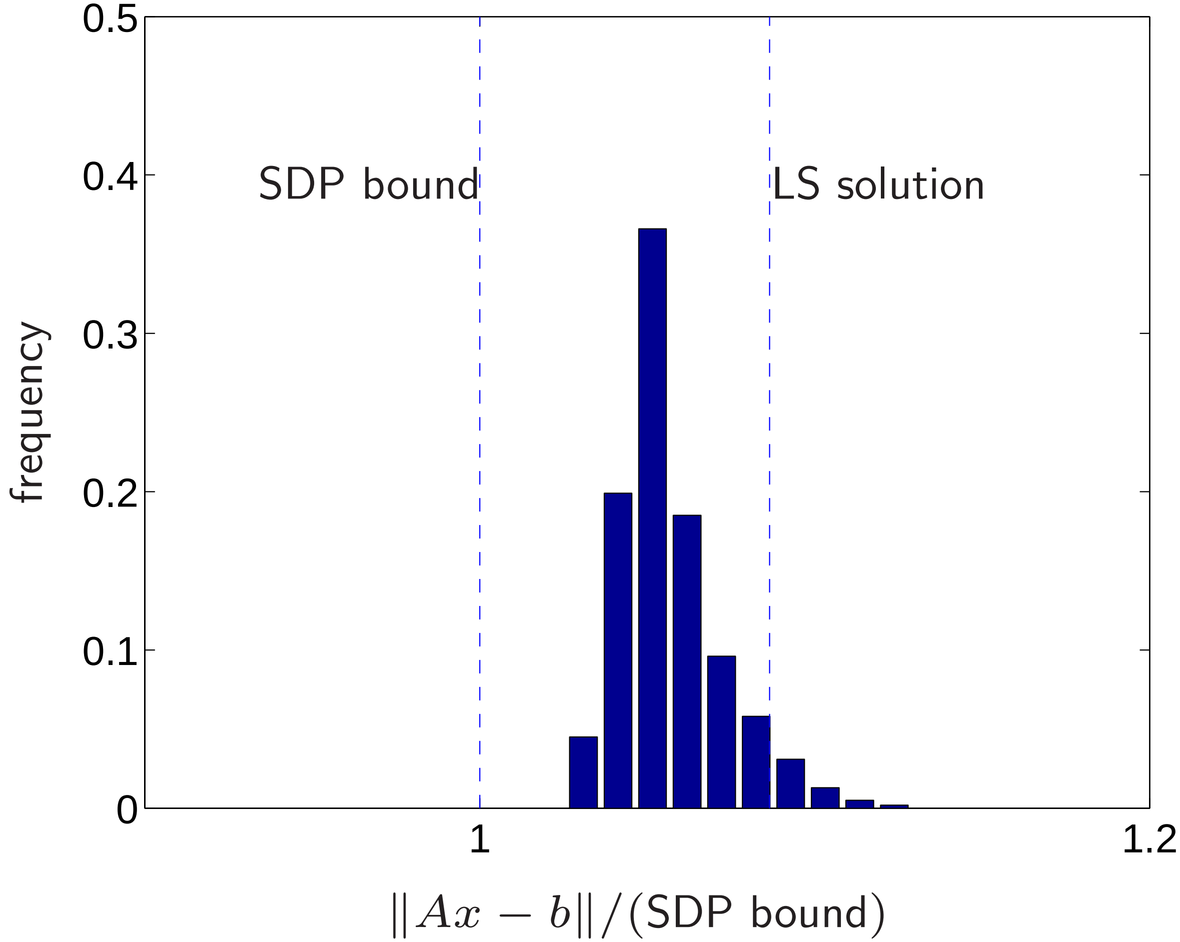
Example

- (randomly chosen) parameters $A \in \mathbf{R}^{150 \times 100}$, $b \in \mathbf{R}^{150}$
- $x \in \mathbf{R}^{100}$, so feasible set has $2^{100} \approx 10^{30}$ points

LS approximate solution: minimize $\|Ax - b\|$ s.t. $\|x\|^2 = n$, then round yields objective 8.7% over SDP relaxation bound

randomized method: (using SDP optimal distribution)

- best of 20 samples: 3.1% over SDP bound
- best of 1000 samples: 2.6% over SDP bound

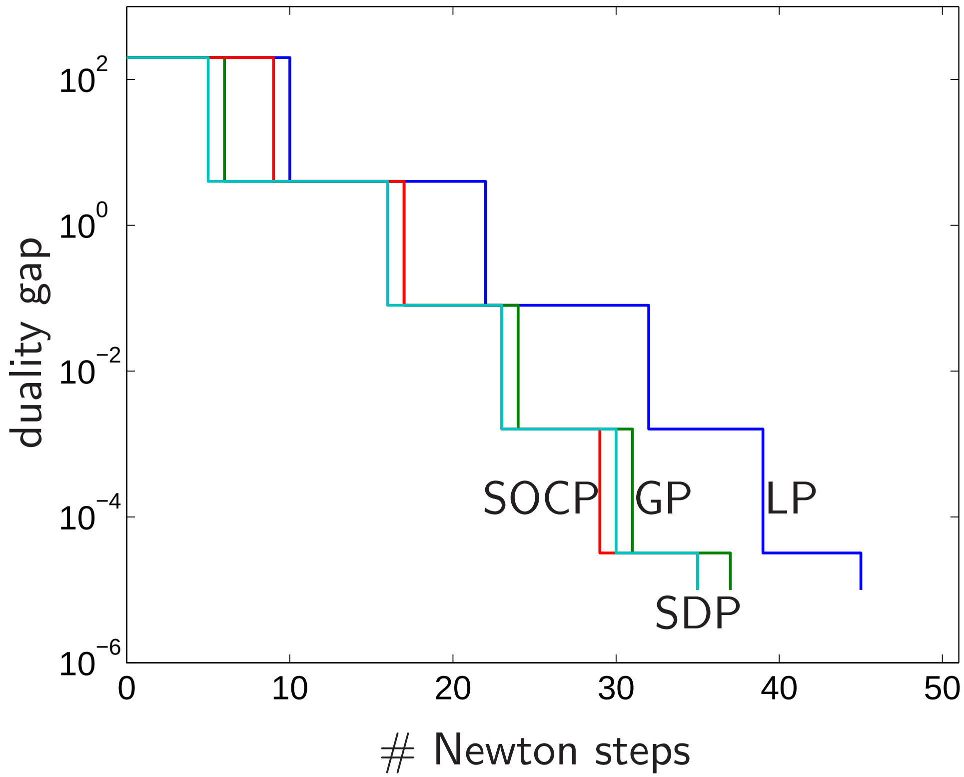

0.5
0.4
0.3
0.2
0.1
0
frequency
SDP bound
LS solution
1
1.2
$\|Ax - b\|/(\text{SDP bound})$

Interior-Point Methods

Interior-point methods

- handle linear and **nonlinear** convex problems *Nesterov & Nemirovsky*
- based on Newton's method applied to 'barrier' functions that trap x in **interior** of feasible region (hence the name IP)
- worst-case complexity theory: # Newton steps $\sim \sqrt{\text{problem size}}$
- in practice: # Newton steps between 10 & 50 (!)
  — over wide range of problem dimensions, type, and data
- 1000 variables, 10000 constraints feasible on PC; far larger if structure is exploited
- readily available (commercial and noncommercial) packages

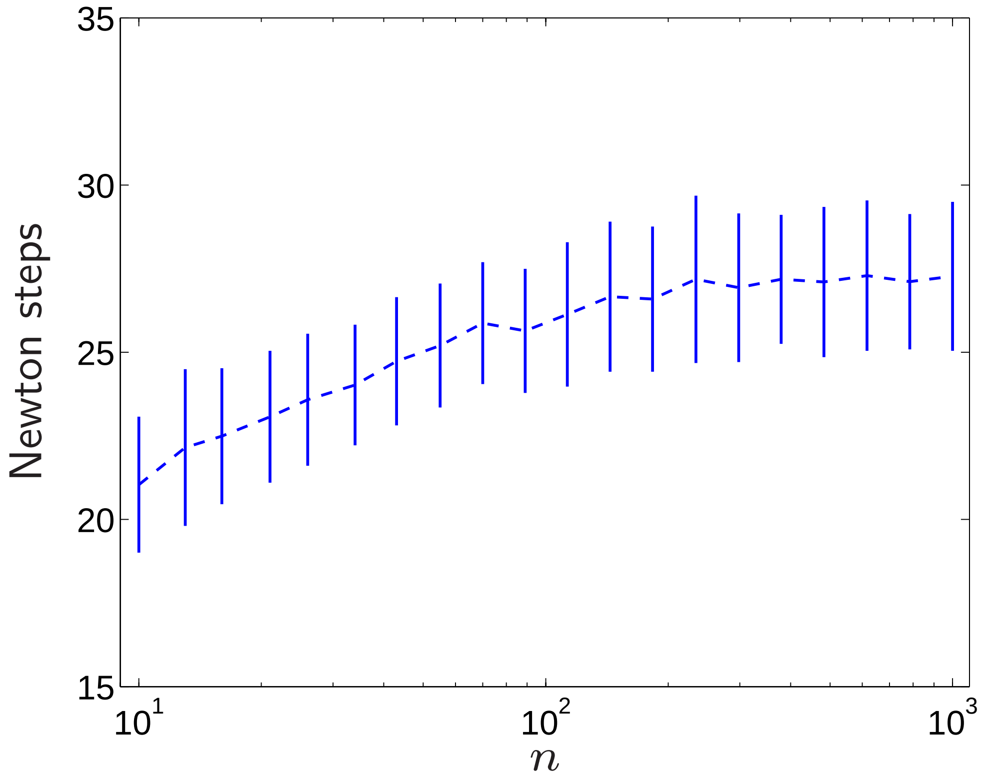
Typical convergence of IP method



LP, GP, SOCP, SDP with 100 variables

Typical effort versus problem dimensions

- LPs with n vbles, $2n$ constraints
- 100 instances for each of 20 problem sizes
- avg & std dev shown



Computational effort per Newton step

- Newton step effort dominated by solving linear equations to find primal-dual search direction
- equations inherit structure from underlying problem
- equations same as for least-squares problem of similar size and structure

conclusion:

we can solve a **convex problem** with about the same effort as solving **30 least-squares problems**

Problem structure

common types of structure:

- sparsity
- state structure
- Toeplitz, circulant, Hankel; displacement rank
- Kronecker, Lyapunov structure
- symmetry

Exploiting sparsity

- well developed, since late 1970s
- direct (sparse factorizations) and iterative methods (CG, LSQR)
- standard in general purpose LP, QP, GP, SOCP implementations
- can solve problems with 10^5, 10^6 vbles, constraints (depending on sparsity pattern)

Exploiting structure in SDPs

in **combinatorial optimization**, major effort to exploit structure

- structure is mostly (extreme) sparsity
- IP methods and others (bundle methods) used
- problems with 10000×10000 LMIs, 10000 variables can be solved

Ye, Wolkowicz, Burer, Monteiro . . .

Exploiting structure in SDPs

in **control**

- structure includes sparsity, Kronecker/Lyapunov
- substantial improvements in order, for particular problem classes

Balakrishnan & Vandenberghe, Hansson, Megretski, Parrilo, Rotea, Smith, Vandenberghe & Boyd, Van Dooren, . . .

. . . but **no general solution yet**

Conclusions

Conclusions

convex optimization

- theory fairly mature; practice has advanced tremendously last decade
- qualitatively different from general nonlinear programming
- becoming a **technology** like LS, LP (esp., new problem classes), reliable enough for embedded applications
- cost only $30\times$ more than least-squares, but far more expressive
- **lots of applications** still to be discovered

Some references

- Semidefinite Programming, *SIAM Review 1996*
- Applications of Second-order Cone Programming, *LAA 1999*
- Linear Matrix Inequalities in System and Control Theory, *SIAM 1994*
- Interior-point Polynomial Algorithms in Convex Programming, *SIAM 1994, Nesterov & Nemirovsky*
- Lectures on Modern Convex Optimization, *SIAM 2001, Ben Tal & Nemirovsky*

Shameless promotion

Convex Optimization, *Boyd & Vandenberghe*

- to be published 2003
- good draft available at Stanford EE364 (UCLA EE236B) class web site as course reader