

for computational purposes, because they provide independent bounds on the optimal value. The remaining duals are useful for sensitivity analysis. The superadditive and branch and bound duals provide a more complete analysis of right-hand side sensitivity. The latter requires a branch and bound solution of the problem but considerably less computation. Inference duality requires a branch and bound solution and provides only sensitivity ranges, but ranges can be obtained for all problem data by solving linear programming problems.

One can also formulate a dual based on congruence relations [21] that is not discussed here. H.P. Williams provides an interesting discussion of this and some other duals (surrogate, Lagrangian, superadditive) in [22]. General treatments of Lagrangian and superadditive duality may be found in [17,19], with a brief discussion of surrogate duality in the former. Excellent presentations of Lagrangian duality appear in [6] and [1, Chap. 6].

#### See also

- ▶ [Decomposition Techniques for MILP: Lagrangian Relaxation](#)
- ▶ [Integer Programming](#)
- ▶ [Integer Programming: Algebraic Methods](#)
- ▶ [Integer Programming: Branch and Bound Methods](#)
- ▶ [Integer Programming: Branch and Cut Algorithms](#)
- ▶ [Integer Programming: Cutting Plane Algorithms](#)
- ▶ [Integer Programming: Lagrangian Relaxation](#)
- ▶ [Simplicial Pivoting Algorithms for Integer Programming](#)
- ▶ [Time-dependent Traveling Salesman Problem](#)

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## Integer Programming: Lagrangian Relaxation

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## Article Outline

Keywords

Purpose of Relaxation

Lagrangian Relaxation

The Lagrangian Dual

Integer Programming

Solving the Dual

Further Reading and Extensions

See also

References

## Keywords

Lagrangian relaxation; Integer programming; Lagrangian dual; Lagrange multipliers; Branch and bound

Relaxation is important in optimization because it provides bounds on the optimal value of a problem. One of the more popular forms of relaxation is Lagrangian relaxation, which is used in integer programming and elsewhere.

A problem is relaxed by making its constraints weaker, so that the feasible set is larger, or by approximating the objective function. In the case of a minimization problem, the optimal value of the relaxation is a lower bound on the optimal value of the original problem. For a maximization problem it is an upper bound. The art of relaxation is to design a relaxed problem that is easy to solve and yet provides a good bound.

## Purpose of Relaxation

Relaxation bounds are useful for two reasons. First, they can indicate whether a suboptimal solution is close to the optimum. If a minimization problem, for example, is hard to solve, one might settle for a suboptimal solution whose value is close to a known lower bound. An optimal solution would not be much better.

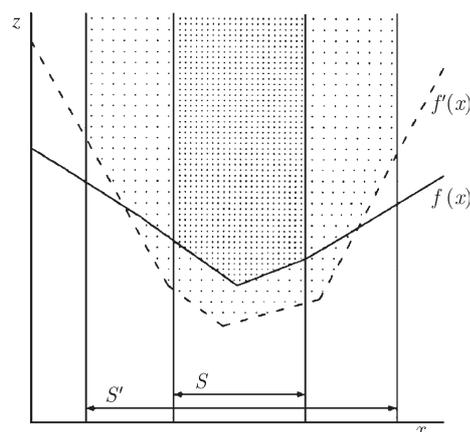
Second, relaxation bounds are useful in accelerating a search for an optimal solution. In a solution of an integer programming problem, for example, one normally solves a relaxation of the problem at each node of the *branch and bound* tree. Suppose again that the objective is to minimize. If the value of the relaxation at some node is greater than or equal to the value of a feasible solution found earlier in the search, then there is no point in branching further at that node. Any optimal solution

found by branching further will have a value no better than that of the relaxation and therefore no better than that of the solution already found. Lagrangian relaxation is often used in this context, because it may provide better bounds than the standard linear programming (LP) relaxation.

## Lagrangian Relaxation

Lagrangian relaxation is named for the French mathematician J.L. Lagrange, presumably due to the occurrence of what we now call Lagrange multipliers in his calculus of variations [2]. Because this form of relaxation changes the objective function as well as enlarging the feasible set, it is necessary to broaden the concept of relaxation somewhat.

Consider the problem of minimizing a function  $f(x)$  subject to  $x \in S$ , where  $x$  is a vector of variables and  $S$  the set of feasible solutions. The *epigraph* of the problem is the set of all points  $(z, x)$  for which  $x \in S$  and  $z \geq f(x)$ . This is illustrated in Fig. 1. The problem of minimizing  $f'(x)$  subject to  $x \in S'$  is a *relaxation* of the original problem if its epigraph contains the epigraph of the original problem. That is, a)  $S \subset S'$  and b)  $f(x) \leq f'(x)$  for all  $x \in S$ . Relaxation is therefore conceived as enlarging the epigraph; enlarging the feasible set is a special case. It is clear that the optimal value of a relaxation still provides a lower bound on the optimal value of the original problem.



Integer Programming: Lagrangian Relaxation, Figure 1  
Epigraph of an optimization problem  $\min \{f(x): x \in S\}$  (darker shaded area) and of a relaxation  $\min \{f'(x): x \in S'\}$  (darker and lighter shaded areas)

Lagrangian relaxation is available for problems in which some of the constraints are inequalities or equations. Such problems may be written as

$$\text{minimize } f(x) \quad (1)$$

$$\text{subject to } g(x) \leq 0 \quad (2)$$

$$x \in S. \quad (3)$$

Here,  $g(x)$  is a vector of functions  $(g_1(x), \dots, g_m(x))$ , and (2) is a family of  $m$  constraints  $g_i(x) \leq 0$ . There is no loss of generality in omitting equality constraints  $h_i(x) = 0$  from this formulation, because they can be written as two inequality constraints,  $h_i(x) \leq 0$  and  $-h_i(x) \leq 0$ . The constraints (3) may take any form, inequality or otherwise.

The *Lagrangian relaxation* is formed by ‘dualizing’ the constraints (2):

$$\begin{cases} \min & f(x) + \lambda g(x) \\ & x \in S. \end{cases} \quad (4)$$

Here,  $\lambda = (\lambda_1, \dots, \lambda_m)$  is a vector of nonnegative *Lagrange multipliers* that correspond to the inequality constraints. The aim of dualization is to remove the hardest constraints from the constraint set, so that the relaxed problem is relatively easy to solve.

The Lagrangian relaxation is in fact a relaxation because its epigraph contains the epigraph of the original problem (1)-(3). This can be verified by checking conditions a) and b):

- The feasible set of the original problem is a subset of the feasible set of the relaxation, because the relaxation omits some of the original constraints.
- If  $x$  is feasible in the original problem, then  $f(x) \geq f(x) + \lambda g(x)$ . This is because  $\lambda \geq 0$  and, due to the feasibility of  $x$ ,  $g(x) \leq 0$ .

### The Lagrangian Dual

A relaxation can be constructed simply by eliminating the constraints (2) rather than dualizing them. One might ask what is the advantage of dualization. One rationale is that when the Lagrange multipliers are properly chosen, the penalties  $\lambda_i g_i(x)$  in the objective function hedge against infeasibility. To the extent that constraints  $g_i(x) \leq 0$  are violated and the bound

thereby weakened, the objective function will be penalized, restoring the quality of the bound.

Fortunately one can search for a proper choice of multipliers. The Lagrangian relaxation is actually a ‘family’ of relaxations, parameterized by the vector  $\lambda$  of multipliers. This provides the possibility of searching over values of  $\lambda$  to find a relaxation that gives a good lower bound on the optimal value.

The problem of finding the best possible relaxation bound is the *Lagrangian dual* problem. If  $\theta(\lambda)$  is the optimal value of the relaxation (4), the Lagrangian dual of (1)-(3) is the problem of maximizing  $\theta(\lambda)$  subject to  $\lambda \geq 0$ .

Under certain conditions the best relaxation bound is equal to the optimal value of the original problem (1)-(3) [1]. Generally, however, it falls short. The amount by which it falls short is the *duality gap*.

The Lagrangian dual problem has three attractive features:

- It need not be solved to optimality. Any feasible solution provides a valid lower bound.
- Its objective function  $\theta(\lambda)$  is always a concave function of  $\lambda$ . One need only find a local maximum, which is necessarily a global maximum as well.
- Its solutions have a *complementary slackness* property. If certain  $\lambda_i$ 's are positive in an optimal solution of the dual problem, then the corresponding constraints  $g_i(x) \leq 0$  are satisfied as equations in some optimal solution of the primal problem (1)-(3).

A serious drawback of the Lagrangian dual is that simply evaluating the objective function  $\theta(\lambda)$  for a given  $\lambda$  normally requires solution of an optimization problem. The relaxation must be carefully chosen so that this is practical. Moreover the function  $\theta$  is typically nondifferentiable.

Why is the Lagrangian dual a ‘dual’? Perhaps because it generalizes the LP dual, which is the Lagrangian dual of an LP problem. To see this, consider the LP problem  $\min \{cx: Ax \geq a, x \geq 0\}$ . Its Lagrangian dual is to maximize

$$\begin{aligned} \theta(\lambda) &= \min_{x \geq 0} \{cx + \lambda(a - Ax)\} \\ &= \min_{x \geq 0} \{(c - \lambda A)x + \lambda a\} \end{aligned}$$

over  $\lambda \geq 0$ . So  $\theta(\lambda)$  is  $-\infty$  if some component of  $c - \lambda A$  is negative and is  $\lambda a$  otherwise. This means that

maximizing  $\theta(\lambda)$  over  $\lambda \geq 0$  is equivalent to maximizing  $\lambda a$  subject to  $\lambda A \leq c$  and  $\lambda \geq 0$ , which is precisely the LP dual.

The duality relationship holds more convincingly, however, between two problem-solving strategies: solution of strengthenings and solution of relaxations [12]. Methods that solve strengthenings include branching methods, local search heuristics, and other techniques that enumerate solutions or partial solutions by fixing some or all of the variables. Solution of each strengthening provides an upper bound on the optimal value, and the goal is to find the smallest upper bound. If the search is exhaustive, the smallest upper bound is equal to the optimal value.

The dual strategy is to solve relaxations of the problem in order to find the largest possible lower bound on the optimal value. There is no obvious way to enumerate relaxations, however, unless they are somehow parametrized, in which case one can enumerate values of the parameters. The Lagrangian dual is one way of doing this but by no means the only. Another is the *surrogate dual* [7,8,9], in which the relaxed constraint set is a nonnegative linear combination of inequality constraints, and relaxations are parametrized by the vector of multipliers in the linear combination. The dual approach also differs from the primal in that an exhaustive enumeration normally does not guarantee that the best bound obtained is equal to the optimal value. There is usually a duality gap.

### Integer Programming

The application of Lagrangian ideas to integer programming dates back at least to H. Everett [4]. In this arena the optimization problem (1)-(3) becomes,

$$\begin{cases} \min & cx \\ \text{s.t.} & Ax \leq a \\ & Bx \leq b \\ & x_j \text{ integer for all } j. \end{cases} \quad (5)$$

The 'hard' constraints  $Ax \leq a$  are dualized in the Lagrangian relaxation,

$$\begin{cases} \min & cx + \lambda(Ax - a) \\ \text{s.t.} & Bx \leq b \\ & x_j \text{ integer for all } j, \end{cases} \quad (6)$$

and  $\theta(\lambda)$  is the minimum value of this problem for a given  $\lambda$ . The optimal value  $z_{LD}$  of the Lagrangian dual is a lower bound on the optimal value  $z_{IP}$  of (5). It will be seen shortly that the bound  $z_{LD}$  is at least as good as the bound  $z_{LP}$  obtained by solving the LP relaxation of (5). (The LP relaxation is the result of dropping the integrality constraints.)

In the context of integer programming, the Lagrangian function  $\theta(\lambda)$  is not only concave but piecewise linear. This is because  $\theta(\lambda)$  is the maximum of a set of linear functions  $cx + \lambda(Ax - a)$  over all integral values of  $x$  that satisfy  $Bx \leq b$ .

A fundamental property of the Lagrangian dual is that  $z_{LD}$  is equal to the optimal value  $z_C$  of

$$\begin{cases} \min & cx \\ \text{s.t.} & Ax \leq a \\ & x \in \text{conv}(S), \end{cases} \quad (7)$$

where  $S$  is the set of integer points satisfying  $Bx \leq b$ , and  $\text{conv}(S)$  is the convex hull of  $S$  [6]. The Lagrangian dual can therefore be written as an LP problem, if a linear description of  $\text{conv}(S)$  is available.

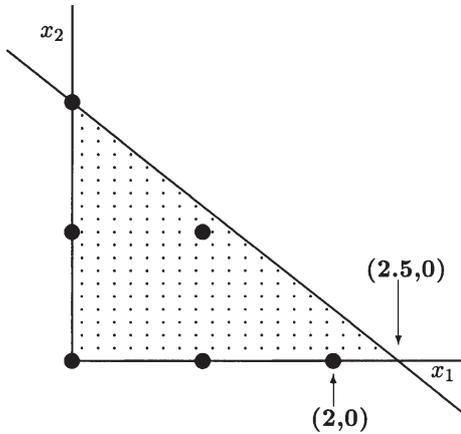
The reasoning behind this fact goes as follows. Because the feasible set of (7) is that of an LP problem, the optimal value of its Lagrangian dual is equal to  $z_C$ . To see that it is also equal to  $z_{LD}$ , thereby proving  $z_C = z_{LD}$ , it suffices to show that the Lagrangian relaxation of (7) always has the same optimal value as the Lagrangian relaxation of (5). But this is true because the former is the same problem as the latter, except that the constraints in former are  $x \in \text{conv}(S)$  and in the latter are  $x \in S$ . This substitution has no effect on the optimal value because the objective function is linear.

It can now be seen that the bound  $z_{LD}$  is always at least as good as  $z_{LP}$ . Let  $C_{IP}$  be the problem (7) corresponding to (5), and let  $C_{LP}$  be the problem (7) corresponding to the LP relaxation of (5).  $C_{LP}$ 's feasible set contains that of  $C_{IP}$ , and its optimal value is therefore less than or equal to  $z_{LD}$ . But because  $C_{LP}$  is identical to (5)'s LP relaxation,  $z_{LP} \leq z_{LD}$ .

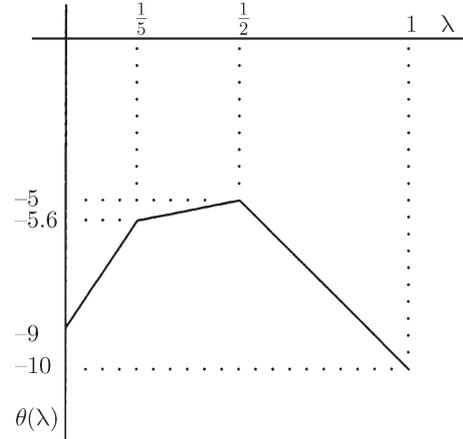
When  $Bx \leq b$  happens to describe a polyhedron whose vertices have integral coordinates,  $C_{IP}$  and  $C_{LP}$  are the same problem. In this case  $z_{LD} = z_{LP}$ .

To sum up,

$$z_{LP} \leq z_{LD} = z_C \leq z_{IP},$$



Integer Programming: Lagrangian Relaxation, Figure 2  
 Feasible set of an integer programming problem (large dots) and its linear programming relaxation (area shaded by small dots). The point (2, 0) is the optimal solution, and (2.5, 0) is the solution of the LP relaxation



Integer Programming: Lagrangian Relaxation, Figure 3  
 The Lagrangian function  $\theta(\lambda)$  for an integer programming problem. The optimal value of the Lagrangian dual problem is  $\theta(1/2) = -5$

where the first inequality is an equation when  $Bx \leq b$  describes an integral polyhedron.

As an example consider the integer programming problem (Fig. 2):

$$\begin{cases} \min & -2x_1 - x_2 \\ \text{s.t.} & 4x_1 + 5x_2 \leq 10 \\ & 0 \leq x_j \leq 3 \\ & x_j \text{ integer, } j = 1, 2. \end{cases} \quad (8)$$

The optimal solution is  $x = (2, 0)$ , with value  $z_{IP} = -4$ . Dualizing the first constraint decouples the variables:

$$\begin{aligned} \theta(\lambda) &= \min_{\substack{0 \leq x_j \leq 3 \\ x_j \text{ integer}}} \{-2x_1 - x_2 + \lambda(4x_1 + 5x_2 - 10)\} \\ &= \min_{\substack{0 \leq x_j \leq 3 \\ x_j \text{ integer}}} \{(4\lambda - 1)x_1 + (5\lambda + 1)x_2 - 10\lambda\}. \end{aligned}$$

Because of the decoupling,  $\theta(\lambda)$  is easily computed:

$$\theta(\lambda) = \begin{cases} 17\lambda - 9 & \text{if } 0 \leq \lambda \leq \frac{1}{5}, \\ 2\lambda - 6 & \text{if } \frac{1}{5} \leq \lambda \leq \frac{1}{2}, \\ -10\lambda & \text{if } \lambda \geq \frac{1}{2}. \end{cases}$$

It is evident in Fig. 3 that  $\theta$  is a concave, piecewise linear function. The optimal value of the Lagrangian dual is  $z_{LD} = \theta(1/2) = -5$ , resulting in a duality gap of  $z_{IP} -$

$z_{LD} = 1$ . The optimal value of the LP relaxation is likewise  $-5$ , so that in the present case  $z_{LP} = z_{LD}$ . This is predictable because  $Bx \leq b$  consists of the bounds  $0 \leq x_j \leq 3$ , which define an integral polyhedron.

In practical applications, the Lagrangian relaxation is generally constructed so that it can be solved in polynomial time. It might be a problem in which the variables can be decoupled, as in the above example, or whose feasible set is an integral polyhedron. Popular relaxations include assignment or transportation problems, which can be solved quickly.

A notable example is the *traveling salesman problem* on  $n$  cities:

$$\text{minimize } \sum_{ij} c_{ij}x_{ij} \quad (9)$$

subject to

$$\sum_j x_{ij} = 1, \quad \text{all } i, \quad (10)$$

$$\sum_i x_{ij} = 1, \quad \text{all } j, \quad (11)$$

$$\sum_{i \notin V} \sum_{j \in V} x_{ij} \geq 1, \quad \text{all nonempty } V \subset \{2, \dots, n\}, \quad (12)$$

$$x_{ij} \geq 0, \quad x_{ij} \text{ integral, all } i, j. \quad (13)$$

If the assignment constraints (10) are dualized, the Lagrangian relaxation minimizes

$$\begin{aligned} \sum_{ij} c_{ij}x_{ij} + \sum_i \lambda_i \left( \sum_j x_{ij} - 1 \right) \\ = \sum_{ij} (c_{ij} + \lambda_i)x_{ij} - \sum_i \lambda_i \end{aligned}$$

subject to (11)-(13). This is equivalent to finding a minimum-cost spanning arborescence that is rooted at node 1, which can be done in polynomial time [3]. Alternatively, the Lagrangian relaxation can be solved as an LP problem without the integrality constraints, because the same optimal value results [3]. See [10,14] for a survey of efforts along this line.

### Solving the Dual

Subgradient optimization is a popular method for solving the Lagrangian dual, because subgradients of  $\theta$  (and gradients when they exist) can be readily calculated.

Let  $X(\bar{\lambda})$  be the set of optimal solutions of the Lagrangian relaxation (4) when  $\lambda = \bar{\lambda}$ . If  $X(\bar{\lambda})$  is a singleton  $\{\bar{x}\}$ , then the gradient of  $\theta$  at  $\bar{\lambda}$  is simply the vector  $g(\bar{x})$ . This is because for values of  $\lambda$  in a neighborhood of  $\bar{\lambda}$ ,  $\theta(\lambda)$  is the linear function  $f(\bar{x}) + \lambda g(\bar{x})$ .

More generally, for any  $\bar{x} \in X(\bar{\lambda})$ ,  $g(\bar{x})$  is a subgradient of  $\theta$  at  $\bar{\lambda}$ . In fact, every subgradient of  $\theta$  at  $\bar{\lambda}$  is a convex combination of subgradients that correspond to the solutions in  $X(\bar{\lambda})$ .

In the integer programming case, the subgradients of  $\theta$  at  $\bar{\lambda}$  are  $A\bar{x} - a$  for each  $\bar{x} \in X(\bar{\lambda})$ , and convex combinations thereof. Consider the example (8), where

$$X(\lambda) = \begin{cases} \{(3, 3)\} & \text{if } 0 \leq \lambda < 1/5, \\ \{(3, 3), (3, 0)\} & \text{if } \lambda = 1/5, \\ \{(3, 0)\} & \text{if } 1/4 < \lambda < 1/2, \\ \{(3, 0), (0, 0)\} & \text{if } \lambda = 1/2, \\ \{(0, 0)\} & \text{if } \lambda > 1/2. \end{cases}$$

Thus at  $\lambda = 0$ ,  $\theta$  has the gradient (slope) of  $4(3) + 5(3) - 10 = 17$ . At  $\lambda = 1/5$ , the subgradients of  $\theta$  are 17, 2, and their convex combinations; i. e., all slopes in the interval  $[2, 17]$ . This can be seen in Fig. 3.

### Further Reading and Extensions

A lucid geometrical exposition of Lagrangian duality may be found in [1, Chap. 6]. A widely read treatment

of its application to integer programming is [5]. A recent tutorial is [11], which also surveys methods for solving the dual. [13, Sect. III.2.6] describes some methods for strengthening the Lagrangian relaxation. There is a vast literature on applications and enhancements.

The idea of the Lagrangian dual need not be limited to the use of Lagrange multipliers. A dual problem can be solved over any parametrized family of relaxations. The dual problem might be solved by a local search heuristic over the parameter space.

### See also

- ▶ [Branch and Price: Integer Programming with Column Generation](#)
- ▶ [Decomposition Techniques for MILP: Lagrangian Relaxation](#)
- ▶ [Integer Linear Complementary Problem](#)
- ▶ [Integer Programming](#)
- ▶ [Integer Programming: Algebraic Methods](#)
- ▶ [Integer Programming: Branch and Bound Methods](#)
- ▶ [Integer Programming: Branch and Cut Algorithms](#)
- ▶ [Integer Programming: Cutting Plane Algorithms](#)
- ▶ [Integer Programming Duality](#)
- ▶ [Lagrange, Joseph-Louis](#)
- ▶ [Lagrangian Multipliers Methods for Convex Programming](#)
- ▶ [LCP: Pardalos-Rosen Mixed Integer Formulation](#)
- ▶ [Mixed Integer Classification Problems](#)
- ▶ [Multi-objective Integer Linear Programming](#)
- ▶ [Multi-objective Mixed Integer Programming](#)
- ▶ [Multi-objective Optimization: Lagrange Duality](#)
- ▶ [Multiparametric Mixed Integer Linear Programming](#)
- ▶ [Parametric Mixed Integer Nonlinear Optimization](#)
- ▶ [Set Covering, Packing and Partitioning Problems](#)
- ▶ [Simplicial Pivoting Algorithms for Integer Programming](#)
- ▶ [Stochastic Integer Programming: Continuity, Stability, Rates of Convergence](#)
- ▶ [Stochastic Integer Programs](#)
- ▶ [Time-dependent Traveling Salesman Problem](#)

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## Integrated Planning and Scheduling

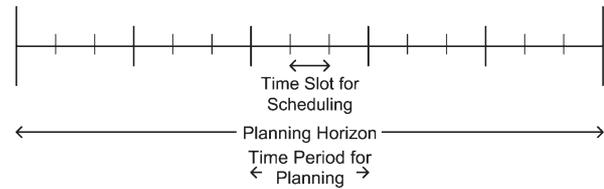
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### Article Outline

[Introduction](#)  
[Representation of Time Slots](#)  
[Problem Definition](#)  
[Solution Strategy](#)  
[Conclusions](#)  
[References](#)

### Introduction

Integrated planning and scheduling of events in production systems are among the most important fac-



**Integrated Planning and Scheduling, Figure 1**  
Time relation between planning and scheduling

tors that affect the efficient operation of these systems. The main objective of integrated planning and scheduling is the allocation of resources that change dynamically over the time domain, and the coordination of the activities that are required to satisfy customer demand. One difference between planning and scheduling is the time scale over which the actions for these factors are triggered. During the planning phase, allocation of available resources and satisfaction of the given demand are made for a medium-term horizon that is usually expressed in months. During the scheduling phase, however, short-term allocation of available resources and the timing of the production of specific orders are the main decisions and these involve time scales of days to weeks.

During both planning and scheduling phases, the time horizon to be planned is divided into time slots. The main difference between planning and scheduling problems in terms of execution times is the length of time scales. Whereas the time scale for the planning phase is longer, the time scales for the scheduling phase are shorter as shown in Fig. 1.

For a realistic case, the time period for the planning phase is measured in weeks or months, therefore, total production and the inventory of each product at the end of each time period are the only performance issues for the system. For scheduling, however, the length of the time slots is measured in hours. Therefore in addition to the production quantity, the production sequence of each product becomes important during scheduling.

Traditionally planning and scheduling have been performed separately on the shop floor, but this decomposition of two activities leads to decreased efficiency of the operations performed in the production centers. Moreover, medium-term plans may result in infeasible projections if they are made without consideration of